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D. E. Littlewood

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INVARIANT THEORY, TENSORS AND GROUP CHARACTERS

By D. E. LITTLEWOOD, *University College, Swansea*

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In Part I is developed the theory of the *tensor* as a device for the construction of concomitants. This part includes the specific separation of a complete cogredient tensor of rank r into simple tensors, with a formula indicating the corresponding separation of a mixed tensor; also the corresponding theory in tensors to the Clebsch theory of algebraic forms, and a compact proof of the fundamental theorem that all concomitants under the full linear group can be obtained by the multiplication and contraction of tensors. The general equivalence is demonstrated, so far as elementary applications are concerned, of the method of tensors with the classical symbolic method of invariant theory.

The first part forms a foundation for the principal theory of the paper which is developed in Part II. This primarily consists of an analysis of the properties of S-functions which provides methods for predicting the exact number of linearly independent concomitants of each type, of a given set of ground forms. Complementary to this, a method of substitutional analysis based on the tableaux which must be constructed in obtaining a product of S-functions, enables the specific concomitants of each type to be written down.

Part III consists of applications to the classical problems of invariant theory. For ternary perpetuants a generating function is obtained which is not only simpler than that given by Young, but is also more general, in so far as it indicates, as well as the covariants, also the mixed concomitants. Extension is made to any number of variables. The complete sets of concomitants, up to degree 5 or 6 in the coefficients, are obtained for the cubic, quartic and quadratic complex in any number of variables. *Alternating concomitant types* are described and enumerated. A *theorem of conjugates* is proved which associates the concomitants of one ground form with the concomitants of a ground form of a different type, namely, that which corresponds to the conjugate partition.

Some indication is made of the extension of this theory to invariants under restricted groups of transformations, e.g. the orthogonal group, but the full development of this extended theory is to be the subject of another paper.

PART I. GENERAL THEORY

Introduction

In algebraic geometry co-ordinates represent a relation between a point or some other geometric entity, and a certain frame of reference. Equations thus represent relations between geometric configurations and this frame of reference. Since it is usually the geometric configuration rather than the frame of reference in which we are interested, it is of considerable advantage to be able to 'eliminate' the frame of reference from these equations.

Given a set of ground forms which correspond to known geometric configurations, it is found that certain functions of the coefficients of these ground forms, and of the variables, are unchanged in form when a transformation is made from one frame of reference to another. These are the *concomitants* of the given set of ground forms. To work with these concomitants is equivalent to the 'elimination' of the frame of reference. Theoretically at least, all properties of the geometric configuration as distinct from the frame of reference, are expressible in terms of these concomitants.

The advantages of such a method of investigation are so obvious, that it is not surprising that, during the last half of the nineteenth century and at the beginning of this century, a very considerable volume of study was made on these lines by Cayley, Sylvester, Clebsch, Gordan, Hilbert and others, which is described under the general title of Invariant Theory.

Of recent years interest in invariant theory has flagged somewhat, and perhaps, setting aside the vagaries of fashion, one reason for this has been the introduction of *tensors*.

The *tensor* is, in essence, an algebraic device for the construction of the concomitants of a given set of ground forms. The tensor notation is simple, the rules for the construction of concomitants are easily mastered, and the device thus presents the superficial advantages whilst avoiding the difficulties of theory and technique, of classical invariant theory.

The most effective work in classical invariant theory is based on the symbolic method due to Aronhold. It will be shown in this paper that, in its elementary application, the uses of tensors is mathematically equivalent to the use of Aronhold symbols. By either method an unlimited variety of expressions may be written down which are guaranteed to have the property of concomitants. There is no corresponding guarantee, however, except in the detailed evaluation, that any given expression will not prove to be identically equal to zero, or that two different expressions, when evaluated, will not prove to be one and the same.

The construction of the linearly independent or algebraically independent concomitants by either method requires the development of a special technique, and a large body of the work in classical invariant theory is concerned with this technique. Little or no corresponding technique has previously been developed for tensors. Indeed, it has not previously been proved that every concomitant can be expressed in terms of tensors.

Although the method of tensors is equivalent to the symbolic method in its elementary application, because of the difference in form, the appropriate technique develops quite differently. The manipulation of the symbols has called forth a technique based on the processes of ordinary algebra, but the manipulation of the suffixes of the tensors finds a more fitting technique in group theory and in the Quantitative Substitutional Analysis as developed by Young.*

It is the purpose of this paper to co-ordinate these two methods, the quantitative substitutional methods of Young, and a new method mentioned in a previous paper (Littlewood 1936*b*) involving a so-called 'new multiplication of *S*-functions'. After the proof of the fundamental theorem that all concomitants can be obtained by contraction from the basic tensors, tensor variables and certain fundamental tensors, the equivalence of the method with the use of Aronhold symbols is indicated. Part I concludes with some indication of the modifications necessary when the group of transformations is a restricted group such as the orthogonal group.

In Part II group representational methods are developed based on the 'new multiplication of *S*-functions' and on substitutional analysis. In Part III the theory is described by applications to the classical problems of invariant theory.

Derived transformations

Suppose that a set of n variables $x^1, x^2, \ldots, x^n$ are transformed by a linear substitution

$$x'^i = \Sigma \xi^i_j x^j.$$

It should be noted that the numbers 1, 2, …, n, and the letters i, j are upper suffixes such as are used in tensor calculus, and not indices.

* Nevertheless, historically, it was from the symbolic method that Young discovered his Quantitative Substitutional Analysis (1901–35).

It is convenient to be able to express the linear transformation in matrix form. Forming the variables into column vectors $X = [x^s]$, $X' = [x'^s]$ and putting $A = [\xi^s_t]$, then

$$X' = AX.$$

In writing matrices the letters s and t are reserved to denote the indices of the row and column respectively of a typical element. The sign $\sim$ placed above a matrix indicates the transposition of rows and columns. Thus $\tilde{X}$ is the row vector $[X^t]$, and $\tilde{A} = [\xi^t_s]$. The sign is usually placed above a letter which denotes a row vector.

Suppose that while X is undergoing this transformation the bilinear form

$$\tilde{U}X = [u_t][x^s] = \Sigma u_r x^r$$

remains invariant. This implies that the variables u_r must undergo the reciprocal transformation

$$U' = A^{-1}U, \quad \text{or} \quad u'_i = \Sigma \eta^j_i u_j,$$

where

$$[\eta^s_t][\xi^s_t] = I,$$

the unit matrix.

More generally it may be supposed that the set of variables $x^1, \ldots, x^n$ undergo the given transformation, and that $v_1, v_2, \ldots, v_N$ is another set of variables with N not necessarily equal to n. Then if $f(x^i, v_j)$ is some function of both sets of variables, it may happen that the form $f(x^i, v_j)$ can be kept invariant if and only if the variables are subjected simultaneously to a transformation which is uniquely determined by the transformation of the x^i. Then the v_j's will be called a *set of derived variables*, which are subjected to a *derived transformation*. The matrix of this transformation will be called a *derived matrix*.

Thus if the quadratic form $\Sigma a_{ij} x^i x^j$ is kept invariant, then the coefficients a_{ij} undergo a derived transformation. If a polynomial of degree p, $\Sigma a_{i_1 i_2 \ldots i_p} x^{i_1} \ldots x^{i_p}$ is kept invariant, the matrix of transformation of the coefficients is usually called the pth induced matrix of A^{-1}. Thus induced matrices are special cases of derived matrices.

The possibility of deriving a matrix in this way is delegated to derived variables. Thus if $y^j, z^k, u_p, v_q, \ldots$ are sets of derived variables, then the invariance of a function

$$f(x^i, y^j, z^k, u_p, v_q, \ldots, w_r)$$

may imply a unique transformation of the last set, w_r, and this will be a derived transformation.

Thus derivation is a property of a function (or functions) f which becomes an absolute invariant when the appropriate transformation is made on the introduced set of variables.

Let A be the matrix of transformation of the x^i, and let $T(A)$ denote any derived matrix. As a result of two consecutive transformations of the x^i with matrices A and B respectively, a transformation is obtained with matrix BA. Corresponding to BA the derived matrix $T(BA)$ could be obtained as the product of the derived matrices $T(B)\,T(A)$ and thus

$$T(BA) = T(B)\,T(A).$$

This is the equation used by Schur (1901) in his famous inaugural dissertation to define an *invariant matrix*. Schur makes the assumption that the elements of the matrix $T(A)$ are polynomials in the elements of A. Making a similar assumption, it will follow that a derived matrix is an invariant matrix according to Schur's definition.

This restriction is, however, too stringent for the present purpose. It would exclude even the reciprocal matrix. It will be assumed instead that the elements of $T(A)$ are rational functions of the elements of A.

Now assume that the transformation is non-singular, so that the determinant $|A|$ is not zero. Apart from this the elements of A are in no way restricted and may take values over the whole range of complex numbers. Now if A is a finite matrix $T(A)$ cannot have infinite elements. But if a rational element of $T(A)$ has in its denominator any factor which is not a power of $|A|$, A could be chosen so that this factor was zero, and $T(A)$ would have an infinite element. It follows that the only denominators that can occur in $T(A)$ are powers of the determinant $|A|$.

It is thus seen that the only modification of Schur's theory which follows from this generalization is the introduction of possible negative powers of the determinant of the original matrix.

Hence a derived matrix is either an invariant matrix, or an invariant matrix multiplied by a negative power of the determinant of the transformation.

Definition is now given to special sets of derived variables called *tensors*. Since it will be shown that a tensor can be formed corresponding to each of Schur's invariant matrices, it will follow that every set of derived matrices can be represented as a tensor.

Cogredient and contragredient variables

There is a slight difference in the phraseology of invariant theory and tensor calculus. In invariant theory, two sets of variables are said to be *cogredient* if they undergo the same linear transformation, and *contragredient* if they undergo reciprocal transformations. The words *cogredient* and *contragredient* have a relative meaning. In tensor calculus the meaning of the words is made absolute by comparing all sets of variables with one given set. That this absolute meaning is sometimes assumed in invariant theory is apparent from the words covariant and contravariant, when the comparison is assumed to be with the original ground form.

Making use of upper and lower suffixes it is convenient to adopt here the conventions of tensor calculus. The set of variables x^i as the basis for comparison is not used, but a set u_j to which the variables x^i are contragradient.

Assuming that the variables x^i, u_j are subject to the restriction that the form $\Sigma x^i u_i$ is kept invariant, the variables u_i will be called *cogredient* and the variables x^i *contragredient*. Lower suffixes will be used for cogredient and upper suffixes for contragredient sets of variables, or tensors.

An algebraic form will be called a *covariant* or *contravariant* according as the set of coefficients (*not* the variables) is transformed as a cogredient or contragredient tensor. Since the coefficients transform reciprocally to the variables a covariant will be of the form $f(a, x^i)$, and a contravariant $f(a, u_j)$. More generally if several sets of contragredient variables are involved, e.g. $f(a, x^i, y^j, \ldots)$, the form is called *covariantive*; and if several sets of cogredient variables, e.g. $f(a, u_i, v_j, \ldots)$, it is called *contravariantive*. This conforms to usual practice in invariant theory, and the variations thus introduced in the accepted terminology of invariant theory are as slight as possible.

Simple and complex sets of derived variables

Let y_i $(1 \leqslant i \leqslant m)$ represent a set of derived variables. Then if z_i $(1 \leqslant i \leqslant m)$ is a set of variables obtained from the y_i by a fixed linear substitution (i.e. one that is independent of the group of transformations), the z_i also form a set of derived variables. This is easily seen if, in the invariant function which defines the y_i as derived variables, for each y_i is substituted the corresponding linear function of the z_i. The two sets of variables are said to be *equivalent*.

If a set of derived variables separates into two sets, the variables of each set being transformed amongst themselves only, by the derived transformation, the complete set is said to be *complex*. A set which is equivalent to a complex set is also said to be complex. A set which is not complex is *simple*. Schur uses the words *reducible* and *irreducible* (reduktibel, irreduktibel) in this sense, but since these words have an entirely different significance in invariant theory, the words *complex* and *simple* are preferred.

The expression of a complex set of derived variables as a sum of simple sets thus follows exactly Schur's reduction of the reducible invariant matrix into a direct sum of irreducible invariant matrices.

A simple set of derived variables is a set such that the matrix of transformation is an irreducible invariant matrix.

Tensors

If $x^i_{(1)}, x^i_{(2)}, \ldots, x^i_{(q)}$ represent q sets each of n contragredient variables, and $u^{(1)}_j, u^{(2)}_j, \ldots, u^{(p)}_j$ represent p sets of cogredient variables, then a set of n^{p+q} derived variables can be defined by holding invariant the form

$$\Sigma A^{i_1 \ldots i_p}_{j_1 \ldots j_q} x^{j_1}_{(1)} x^{j_2}_{(2)} \ldots x^{j_q}_{(q)} u^{(1)}_{i_1} u^{(2)}_{i_2} \ldots u^{(p)}_{i_p}.$$

The suffixes enclosed in brackets are distinguishing marks only, and are not subject to the conventions associated with the tensor upper and lower suffixes.

Such a set of n^{p+q} variables $A^{i_1 \ldots i_p}_{j_1 \ldots j_q}$ is called a *complete tensor of rank* $(p+q)$.

If the cogredient variables are subjected to the transformation

$$u'_i = \Sigma \xi^j_i u_j \quad \text{with} \quad u_j = \Sigma \eta^i_j u_i{}',$$

then also

$$x^j = \Sigma \xi^j_i x'^{\,i}.$$

By comparing the invariant form with its form after transformation the following equation gives the manner in which a tensor transforms:

$$A'^{\,i_1' \ldots i_p'}_{j_1' \ldots j_q'} = \Sigma \eta^{i_1'}_{i_1} \ldots \eta^{i_p'}_{i_p} \xi^{j_1}_{j_1'} \ldots \xi^{j_q}_{j_q'} A^{i_1 \ldots i_p}_{j_1 \ldots j_q},$$

the summation being with respect to all repeated suffixes.

This equation giving the manner of transformation may be taken as an alternative definition of a tensor.

Two tensors are said to be *equivalent* if they are equivalent sets of derived variables. A tensor is said to be *complex* or simple according as it is a complex or simple set of derived variables. A complete tensor of rank greater than one is in general complex. The n^{p+q} linearly independent terms of a complete tensor of rank $(p+q)$ may thus have a subset of terms which forms a set of derived variables. Such a subset is called a tensor of rank $(p+q)$.

If a tensor is complex and separates into two parts, one of the parts (simple or complex) will be called a *subtensor*.

A complete tensor of rank $(p+q)$ is thus expressible as a sum of simple tensors of rank $(p+q)$. Thus the complete cogredient tensor of rank 2, A_{ij}, is a sum of two simple tensors of orders respectively $\frac{1}{2}n(n+1)$ and $\frac{1}{2}n(n-1)$.

If
$$B_{ij} = \tfrac{1}{2}(A_{ij}+A_{ji}), \quad C_{ij} = \tfrac{1}{2}(A_{ij}-A_{ji}),$$
then
$$A_{ij} = B_{ij}+C_{ij}.$$
The tensor B_{ij} is called a symmetric tensor and satisfies
$$B_{ij} = B_{ji},$$
while C_{ij} is called antisymmetric and satisfies
$$C_{ij} = -C_{ji}.$$

Later it is shown that a complete cogredient tensor of rank 3 has four simple components. Put
$$A_{ijk} = B_{ijk}+C_{ijk}+D_{ijk},$$
where
$$6B_{ijk} = A_{ijk}+A_{jki}+A_{kij}+A_{ikj}+A_{kji}+A_{jik},$$
$$6C_{ijk} = A_{ijk}+A_{jki}+A_{kij}-A_{ikj}-A_{kji}-A_{jik},$$
$$3D_{ijk} = 2A_{ijk}-A_{jki}-A_{kij}.$$

Of these B_{ijk} is a symmetric tensor of order $\frac{1}{6}n(n+1)(n+2)$, and C_{ijk} is an antisymmetric tensor of order $\frac{1}{6}n(n-1)(n-2)$. The tensor D_{ijk} is not simple, but is equivalent to two simple tensors. The separation into two subtensors can be made in an infinity of ways. Each simple subtensor of D_{ijk} has $\frac{1}{3}n(n^2-1)$ terms.

The simple tensors can be classified according to the matrix of transformation. This classification is straightforward in the case of cogredient tensors and follows Schur's theory of invariant matrices (Schur 1901; see also Littlewood 1940, chapter x).

The spur of a matrix $[a_{st}]$ is defined as the sum of the diagonal elements Σa_{ii}. As will be seen from the equation for the method of transformation of a tensor, the matrix of transformation for the complete cogredient tensor of rank r is equivalent to the direct product of r matrices each identical with the original matrix. Thus the spur of the matrix of transformation of the complete cogredient tensor of rank r is the rth power of the spur of the original matrix.

If a_r, h_r denote symmetric functions of the latent roots of the original matrix, and $\{\lambda\}\equiv\{\lambda_1, \ldots, \lambda_p\}$ denotes an S-function of these latent roots, then corresponding to each partition (λ) of r into not more than n parts, there is a Schur invariant matrix of which the elements are of degree r in the elements of the original matrix. The spur of this invariant matrix is the S-function $\{\lambda\}$ of the latent roots (Schur 1901; Littlewood 1940).

Hence in order to discover the manner in which the complete cogredient tensor of rank r separates into simple tensors, we have only to express its spur h_1^r as a sum of S-functions. This can be accomplished by the known formula (Littlewood 1940)
$$h_1^r = S_1^r = \Sigma\chi_0^{(\lambda)}\{\lambda\},$$
where $\chi_0^{(\lambda)}$ is the characteristic of the identical element of the symmetric group of order $r!$.

Thus for $r = 3$ we have

$$h_1^3 = \{3\}+2\{21\}+\{1^3\},$$

which equation conforms with, and lies behind, the results of an example already given.

The S-function $\{r\} = h_r$, corresponding to a partition into one part only, is the spur of the rth induced matrix and corresponds to the symmetric tensor. The S-function $\{1^r\} = a_r$ is the spur of the rth compound matrix, and corresponds to the antisymmetric tensor.

The separation of a complete cogredient tensor

From a tensor of rank r, $r!$ equivalent tensors are obtained by writing the suffixes in a different order. Operators are dealt with which permute the suffixes and form linear combinations of the tensors with permuted suffixes. These operators are as used by Young in his quantitative substitutional analysis, although he did not operate upon tensors. The algebra of these operators is the Frobenius algebra of the symmetric group of order $r!$ (Littlewood 1940).

A tensor which is equivalent to a given tensor, or to a subtensor thereof, may be obtained from it by operating upon it with a substitutional operator which permutes the suffixes.

For simplicity this theorem will be proved for a tensor of rank 4, but the manner of proof is quite general and in no way depends on the rank.

Let A_{pqrs} be a tensor, and let B_{pqrs} be either equivalent to it or to a subtensor.

Suppose that
$$B_{pqrs} = \Sigma K_{pqrs}^{p'q'r's'} A_{p'q'r's'}.$$

The symbols $K_{pqrs}^{p'q'r's'}$ represent numerical coefficients, the suffixes in the first instance being regarded as distinguishing marks. Nevertheless, because of the manner of transformation of the tensors A_{pqrs}, B_{pqrs}, it is clear that the quantities $K_{pqrs}^{p'q'r's'}$ will transform like a tensor of rank $(4+4)$. At the same time each coefficient $K_{pqrs}^{p'q'r's'}$ will be an invariant, unaltered by the transformation. Thus a series of linear relations between these coefficients is obtained.

When the basic cogredient variables are subjected to the transformation

$$u_1' = \lambda_1 u_1, \quad u_2' = \lambda_2 u_2, \quad u_3' = \lambda_3 u_3, \quad u_4' = \lambda_4 u_4,$$

then
$$B'_{pqrs} = \lambda_p \lambda_q \lambda_r \lambda_s B_{pqrs}, \quad A'_{pqrs} = \lambda_p \lambda_q \lambda_r \lambda_s A_{pqrs},$$

and thus
$$K_{pqrs}^{p'q'r's'} = \lambda_p \lambda_q \lambda_r \lambda_s K_{pqrs}^{p'q'r's'} \lambda_{p'}^{-1} \lambda_{q'}^{-1} \lambda_{r'}^{-1} \lambda_{s'}^{-1}.$$

Hence for any non-zero coefficient $K_{pqrs}^{p'q'r's'}$ the upper suffixes (p', q', r', s') must represent a permutation of the lower suffixes (p, q, r, s).

It is sufficient to consider B_{1234}. This must be expressible in terms of the twenty-four terms obtained from A_{1234} by permuting the suffixes.

If S_i denotes a permutational operator which permutes the suffixes of a tensor, and $K(S_i)$ is a numerical coefficient, then

$$B_{1234} = \Sigma K(S_i) \,.\, S_i A_{1234}.$$

Now consider the transformation

$$u_p' = \Sigma \xi_p^q u_q.$$

Making the corresponding transformations for the tensors B'_{1234} and A'_{1234}, then, since

$$B'_{1234} = \Sigma K(S_i) \, . \, S_i A'_{1234},$$

$$\Sigma \xi_1^p \xi_2^q \xi_3^r \xi_4^s B_{pqrs} = \Sigma K(S_i) \, . \, S_i \xi_1^p \xi_2^q \xi_3^r \xi_4^s A_{pqrs},$$

and picking out the coefficient of $\xi_1^p \xi_2^q \xi_3^r \xi_4^s$, it follows that

$$B_{pqrs} = \Sigma K(S_i) \, . \, S_i A_{pqrs}.$$

This proof is not invalidated if the rank exceeds the number of variables, for one may postulate new variables and subsequently equate them to zero.

It is clear that the *simple* subtensors of A_{pqrs} will be obtained by taking the substitutional expression $\Sigma K(S_r) \, S_r$ to be an *irreducible idempotent* of the Frobenius algebra.* For if ϵ_i is such an irreducible idempotent, and ϕ is any substitutional operator such that $\phi\epsilon_i \neq 0$, then another substitutional operator ψ can be found such that $\psi\phi\epsilon_i = \epsilon_i$. This implies that $\epsilon_i A_{pqrs}$ is equivalent to, or to a subtensor of, $\phi\epsilon_i A_{pqrs}$ for any substitutional operator ϕ for which $\phi\epsilon_i A_{pqrs} \neq 0$. For this to be the case $\epsilon_i A_{pqrs}$ must be a simple tensor.

Suppose now that A_{pqrs} is a complete cogredient tensor of rank 4. To obtain those simple components of A_{pqrs} which correspond to the partition (λ), $f^{(\lambda)} = \chi_0^{(\lambda)}$ irreducible idempotents of the Frobenius algebra are required. Such a set has been obtained by Young (1928; cf. Littlewood 1940) as follows.

Suppose that the symbols $\alpha_1, \alpha_2, \ldots, \alpha_m$ are to be permuted. Corresponding to the partition $(\lambda) \equiv (\lambda_1, \lambda_2, \ldots, \lambda_p)$ of m, form a tableau by placing λ_1 of the symbols in the first row, λ_2 in the second row, and so on, with finally λ_p symbols in the pth row. The λ_i symbols in the ith row must appear in the first i columns. The sum of the operations of the symmetric group of permutations on the symbols of each row is then taken, and these substitutional expressions are multiplied together to form a product P. The order of this multiplication is not significant for, since the different substitutional expressions involve different symbols, they will be commutative with one another.

The same procedure is then taken with the columns, but with this difference, a minus sign is attached to each negative permutation. This product is denoted by N. Then

$$\frac{f^{(\lambda)}}{m!} PN,$$

or alternatively

$$\frac{f^{(\lambda)}}{m!} NP,$$

is a primitive idempotent of the Frobenius algebra.

Corresponding to the $m!$ arrangements of the symbols in a tableau there are $m!$ primitive idempotents, but these are not all independent.

Young has shown that there are exactly $f^{(\lambda)}$ tableaux which he calls *standard* in which the order of the symbols in each row and in each column follows the natural order, or any assigned order. These may form the basal units for a representation of the Frobenius algebra, and may be used to separate the component parts of a tensor of rank m.

* See Littlewood (1940), also cf. Weyl (1939), who describes these operators as 'Young Symmetrizers'.

Illustration is made with reference to the tensor of rank 4 corresponding to the partition (2^2). Examples of such a tensor are, in algebraic geometry, the coefficients of a quadratic complex, and in Riemannian geometry, the Riemann-Christoffel tensor.

There are two standard tableaux, namely,

$$\begin{pmatrix}\alpha_1 & \alpha_2\\ \alpha_3 & \alpha_4\end{pmatrix}, \quad \begin{pmatrix}\alpha_1 & \alpha_3\\ \alpha_2 & \alpha_4\end{pmatrix},$$

and the corresponding irreducible idempotents of the Frobenius algebra are

$$\tfrac{1}{6}\{[1+(\alpha_1\alpha_2)]\,[1+(\alpha_3\alpha_4)]\,[1-(\alpha_1\alpha_3)]\,[1-(\alpha_2\alpha_4)]\},$$

and $$\{[1+(\alpha_1\alpha_3)]\,[1+(\alpha_2\alpha_4)]\,[1-(\alpha_1\alpha_2)]\,[1-(\alpha_3\alpha_4)]\}.$$

Hence two simple tensors of type (2^2) are given by

$$\begin{aligned}12B_{pqrs} = {} & A_{pqrs}+A_{qprs}+A_{pqsr}+A_{qpsr}\\ & -A_{rqps}-A_{rpqs}-A_{sqpr}-A_{spqr}\\ & +A_{rspq}+A_{rsqp}+A_{srpq}+A_{srqp}\\ & -A_{psrq}-A_{qsrp}-A_{prsq}-A_{qrsp},\\ 12C_{pqrs} = {} & A_{pqrs}+A_{rqps}+A_{psrq}+A_{rspq}\\ & -A_{qprs}-A_{qrps}-A_{sprq}-A_{srpq}\\ & +A_{qpsr}+A_{qrsp}+A_{spqr}+A_{srqp}\\ & -A_{pqsr}-A_{rqsp}-A_{psqr}-A_{rsqp}.\end{aligned}$$

The identical relations satisfied by the tensors B_{pqrs} and C_{pqrs} may be found by obtaining operators which, when multiplied on the right by the corresponding irreducible idempotent of the Frobenius algebra, give zero.

The complete set of relations may be obtained from the substitutional operators corresponding to each of the other partitions, i.e. (4), (31), (21^2) and (1^4), and say for B_{pqrs}, from the operator used to define C_{pqrs}. However, the relations as obtained by this method require a great deal of simplification. It is more convenient to write down the corresponding left factors of zero from an inspection of the Young tableau.

Thus the substitutional operator PN obtained from the tableau

$$\begin{pmatrix}\alpha_1 & \alpha_2\\ \alpha_3 & \alpha_4\end{pmatrix}.$$

is considered. Since $$[1-(\alpha_1\alpha_2)]\,[1+(\alpha_1\alpha_2)] = 0,$$

then $$[1-(\alpha_1\alpha_2)]\,PN = 0,$$

from which it follows that

$$B_{pqrs} = B_{qprs}.$$

Similarly, considering the factor of zero $[1-(\alpha_3\alpha_4)]$,

$$B_{pqrs} = B_{pqsr}.$$

Also $$[1-(\alpha_1\alpha_3)\,(\alpha_2\alpha_4)]\,PN = 0,$$

and thus $$B_{pqrs} = B_{rspq}.$$

Lastly, consider that subgroup of the symmetric group on the four symbols, which contains the substitution $1+(\alpha_1\alpha_2\alpha_3)+(\alpha_1\alpha_3\alpha_2)$. This subgroup corresponds to the compound character (Littlewood 1940, chapter IX) $(4)+(31)+(211)+(1^4)$, which does not include (2^2). Hence the sum of the substitutions of this subgroup must form a left factor of zero and thus

$$B_{pqrs}+B_{qrps}+B_{rpqs} = 0.$$

All the relations between the components of the tensor may be deduced from these four.

It may be noticed that these are not the usual formulae connecting the components of the Riemann-Christoffel tensor, and it is clear that the tensor has been differently expressed.

The tensor could have been obtained in the more familiar form if the substitutional operator PN had been replaced either by NP or by NPN. The substitution of NPN for PN gives an equivalent subtensor of the complete tensor, but expresses it in a different form. The substitution of NP for PN gives a slightly different subtensor which is a linear combination of the tensors B_{pqrs} and C_{pqrs}.

In either case, using the tableau $\begin{pmatrix}\alpha_1 & \alpha_3\\ \alpha_2 & \alpha_4\end{pmatrix}$, and representing the tensor obtained by D_{pqrs}, because of the factors $[1-(\alpha_1\alpha_2)]\,[1-(\alpha_3\alpha_4)]$ in N, $[1+(\alpha_1\alpha_2)]$ and $[1+(\alpha_3\alpha_4)]$ are given as left factors of zero, and thus

$$D_{pqrs} = -D_{qprs} = -D_{pqsr} = D_{qpsr}.$$

Just as before $\qquad D_{pqrs} = D_{rspq}, \quad \text{and} \quad D_{pqrs}+D_{qrps}+D_{rpqs} = 0.$

These are the familiar relations between the components of the Riemann-Christoffel tensor.

It may be noticed that if ϕ is a substitutional operator which corresponds to a matrix of rank 1 in the Frobenius algebra, then ϕ defines both a left and a right module, i.e. the linear sets ϕz and $z\phi$, where z is an arbitrary element of the algebra. Scalar multiples of the element ϕ are the only elements common to both modules, and thus the pair of modules define ϕ save for a numerical coefficient.

If $$B_{pqrs} = \phi A_{pqrs},$$

then it is the right module only which determines the components of A_{pqrs} which go to form the tensor B_{pqrs}. The left module determines the manner of expression of these components as a tensor.

The substitutional operators PN and NPN have the same right module, and the tensors $(PN)\,A_{pqrs}$ and $(NPN)\,A_{pqrs}$ are equivalent, but are differently expressed because of the difference in the left module. The tensor $(NP)\,A_{pqrs}$ contains different components, but the left module of NP being the same as that of NPN, the manner of expression as a tensor is similar to $(NPN)\,A_{pqrs}$.

Generalization of the definition of a tensor

The classical definition of a concomitant (Turnbull 1928) does not require the absolute invariance of the form under the group of transformations. It is sufficient if, after transformation, the form is multiplied by a factor which is not zero, and is independent of the coefficients and variables. It is proved that this factor must be of the form Δ^i, where Δ is the determinant of the transformation, and i is a positive, zero or negative integer. Thus

$$f(a', x') = \Delta^i f(a, x).$$

The index i is called the *weight* of the concomitant. To make the sign of the weight conform to current usage Δ must be the determinant of the transformation for *cogredient* variables.

In particular, an invariant for which such a factor Δ^i must be introduced is called a *relative* invariant, in contrast with an *absolute* invariant for which $i = 0$.

Similarly, the definition of a *tensor* may be generalized by allowing an arbitrary power of Δ in the equation of transformation. This will not affect the definition of the tensor as a set of derived variables, for if $u_i^{(p)}$ represents n sets of cogredient variables, and $x_q^{(j)}$ represents n sets of contragredient variables, then the determinants $|u_t^{(s)}|$ and $|x_{(t)}^s|$ are invariants of weight $+1$ and -1 respectively. The introduction of powers of one or the other into the function which is held invariant will introduce the appropriate power Δ^i in the equation of transformation of the tensor. With tensors as with concomitants, the index i is called the *weight*.

The relation between the weight and rank of a tensor

A cogredient tensor of rank n and type $\{1^n\}$, where n is the number of variables, has a single component, and is thus a relative invariant. It is easily seen to be an invariant of weight unity.

Also, since, in n variables,

$$\{\lambda_1+1, \lambda_2+1, \ldots, \lambda_n+1\} = \{1^n\}\{\lambda_1, \lambda_2, \ldots, \lambda_n\},$$

it is clear that the matrix of transformation of a tensor of type $\{\lambda_1+1, \ldots, \lambda_n+1\}$ is the same as the matrix of transformation of a tensor of type $\{\lambda_1, \ldots, \lambda_n\}$ save for a scalar multiplier equal to Δ.

Thus a tensor of type $\{\lambda_1+1, \lambda_2+1, \ldots, \lambda_n+1\}$ and of weight p is equivalent to a tensor of type $\{\lambda_1, \ldots, \lambda_n\}$ and weight $p+1$.

If a cogredient tensor corresponds to a partition for which $\lambda_n \neq 0$, then the rank may be decreased by n provided that the weight is simultaneously increased by one.

This equivalence may be illustrated with an example. Let A_{ijk} be a symmetric tensor. Corresponding to the tableau

$$\begin{pmatrix} i, j, k, r \\ p, q \end{pmatrix},$$

a tensor of type $\{42\}$ is constructed:

$$B_{ijkpqr} = A_{ijk}A_{pqr} - A_{pjk}A_{iqr} - A_{iqk}A_{pjr} + A_{pqk}A_{ijr}.$$

If there are only two variables, the only components of this tensor which are not zero correspond to one of the four combinations

$$\left.\begin{matrix} i=j=1 \\ p=q=2 \end{matrix}\right\}, \quad \left.\begin{matrix} i=q=1 \\ p=j=2 \end{matrix}\right\}, \quad \left.\begin{matrix} p=j=1 \\ i=q=2 \end{matrix}\right\}, \quad \left.\begin{matrix} p=q=1 \\ i=j=2 \end{matrix}\right\},$$

and also $$B_{11k22r} = -B_{21k12r} = -B_{12k21r} = B_{22k11r}.$$

The tensor is thus reduced to a tensor of rank 2, but because of the appropriate transformation factors pertaining to the specified 1st, 2nd, 4th and 5th suffixes, it has a weight equal to 2.

This conforms with the equation, valid for two variables

$$\{42\} = \{2\}\{1^2\}^2.$$

Contragredient tensors

The case of contragredient tensors is similar to that of cogredient tensors. The matrix of transformation for cogredient variables, A, is replaced by A^{-1}, the matrix of transformation for contragredient variables. This involves the introduction of negative powers of $\Delta = |A|$. In addition each coefficient a_r in the characteristic equation is replaced by a_{n-r}, including the cases $r = 0$, $r = n$.

From the formula (Littlewood 1940, p. 89)

$$\{\tilde{\lambda}\} = |a_{\lambda_s - s + t}|,$$

on replacing each a_r by a_{n-r} it is seen that the S-function $\{\lambda\} \equiv \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is replaced by the S-function

$$\{\lambda_1 - \lambda_n, \lambda_1 - \lambda_{n-1}, \ldots, \lambda_1 - \lambda_2, 0\}.$$

Thus the complete contragredient tensor of rank 4 in four variables separates into simple tensors corresponding to the partitions $\{4^3\}$, $\{3^2 2\}$, $\{2^2\}$, $\{21^2\}$ and $\{0\}$. The tensor corresponding to $\{0\}$ is a relative invariant.

If a contragredient tensor is compared with the reciprocal A^{-1} of the original transformation, and its type is $\{\lambda\}$ relative to this reciprocal transformation, it will be said to be of *contragredient type* $\{\lambda\}$.

Thus a tensor of contragredient type $\{\lambda_1, \ldots, \lambda_n\}$ is a tensor of cogredient type

$$\{\lambda_1 - \lambda_n, \lambda_1 - \lambda_{n-1}, \ldots, \lambda_1 - \lambda_2, 0\}$$

and of weight $-\lambda_1$.

Given a tensor of cogredient type $\{\lambda\}$ and weight p, the equivalent tensor of weight zero is of type $\{\lambda_1 + p, \lambda_2 + p, \ldots, \lambda_n + p\}$, provided that $\lambda_n + p \geqslant 0$.

If $\lambda_n + p < 0$ there is no equivalent tensor of weight zero according to previous definitions. It is convenient, however, to define S-functions with negative parts, and to make tensors correspond to partitions with negative parts so as to make the equation

$$\{\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_n + 1\} = \{1^n\}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$$

true in all circumstances. The relations between the rank and weight of a tensor then hold without restrictions of sign. The contragredient rank is for this purpose reckoned as negative.

Then a tensor of contragredient type $\{\lambda_1, \ldots, \lambda_n\}$ will be of cogredient type

$$\{-\lambda_n, -\lambda_{n-1}, \ldots, -\lambda_1\}.$$

Mixed tensors

A complete mixed tensor of rank $(r+s)$ separates into simple tensors in accordance with the formula which expresses $a_1^r a_{n-1}^s$ as a sum of S-functions.

Thus since

$$a_1 a_{n-1} = \{2, 1^{n-2}\} + \{1^n\},$$

a tensor A_q^p of rank $(1+1)$ is the sum of an invariant $\sum_p A_p^p$ and a simple tensor of type $\{2, 1^{n-2}\}$, or $\{1, 0^{n-2}, -1\}$.

The formulae for the analysis of the complete mixed tensor in the general case will not be examined here.

Multiplication and contraction of tensors

By the product of two tensors with say p and q components respectively is meant the set of pq components obtained by multiplying these components in pairs. To write down such a product it is necessary to give only the product of typical elements.

Tensors have two fundamental properties upon which their usefulness depends:

(1) The product of two tensors of ranks respectively $(r+s)$ and $(r'+s')$ is a tensor of rank $[(r+r')+(s+s')]$.

The property is obvious from the definition of a tensor.

(2) If in a tensor of rank $[(r+1)+(s+1)]$ any given upper suffix is put equal to any given lower suffix and the result summed for all values of the pair of suffixes, a tensor of rank $(r+s)$ is obtained.

In proving this result it is sufficient to consider a tensor of rank $(1+1)$. If there are other upper and lower suffixes, these will appear both in the original and in the final tensor. After transformation there will appear for each suffix a corresponding factor ξ^p_q or η^p_q which will give the correct method of transformation for the final tensor.

If A^p_q is a tensor of rank $(1+1)$, it follows that

$$A'^r_s = \Sigma A^p_q \xi^r_p \eta^q_s.$$

Hence if $$B = \Sigma A^p_p,$$

then $$B' = \sum_{pqr} A^p_q \xi^r_p \eta^q_r.$$

But since $$[\eta^s_t]\,[\xi^s_t] = 1,$$

so that $$\Sigma \eta^q_r \xi^r_p = \delta_{pq},$$

then $$B' = \Sigma A^p_p = B,$$

and B is an invariant or a tensor of rank 0.

This process of putting an upper suffix equal to a lower suffix and summing is called *contraction*.

The summation convention

It is convenient when dealing with tensors to omit the summation sign and to make the convention that if the same symbol is used for both an upper and a lower suffix, then a summation is understood to take place in which this symbol takes all values from 1 to n.

The fundamental alternating tensor

Let $$B_{p_1 \ldots p_r} = \phi A_{p_1 \ldots p_r},$$

where ϕ is a substitutional operator, be a tensor of rank r and type $\{1^r\}$. The appropriate operator ϕ corresponds to a Young tableau which consists of the r symbols placed in one column, and thence ϕ is the negative symmetric group of permutations on these symbols. If α, β are any two of these symbols, then ϕ may be expressed with a left-hand factor $[1-(\alpha\beta)]$. Hence

$$[1+(\alpha\beta)]\,\phi = 0, \quad \text{and} \quad B_{p_1 \ldots p_r} = -(\alpha\beta)\,B_{p_1 \ldots p_r}.$$

Thus each component of the tensor is changed in sign if any two suffixes are interchanged, and such a component can only differ from zero if the r suffixes all take distinct values.

In particular, if $E_{p_1 \dots p_n}$ is of type $\{1^n\}$, with n variables, the only non-zero components are those for which the suffixes form a permutation of 1, 2, ..., n, and if

$$E_{12\dots n} = k,$$

then

$$E_{p_1 p_2 \dots p_n} = \pm k,$$

according as this permutation is positive or negative.

Thus, apart from an arbitrary scalar multiplier the tensor of type $\{1^n\}$ is unique and independent of any sets of variables or coefficients. It is thus a *fundamental* tensor and is called the *fundamental alternating tensor* or simply the *alternating tensor*.

The scalar k is a relative invariant of weight unity, as is easily seen from the equation

$$E'_{p_1 \dots p_n} = E_{q_1 \dots q_n} \eta^{q_1}_{p_1} \dots \eta^{q_n}_{p_n}$$
$$= |\eta^s_t| E_{p_1 \dots p_n}.$$

There is a corresponding contragredient tensor

$$E^{p_1 \dots p_n} = 0 \quad \text{if } p_i = p_j$$
$$= k' \quad \text{if } p_1 p_2 \dots p_n \text{ is a positive permutation of } 12 \dots n$$
$$= -k' \quad \text{if a negative permutation.}$$

Clearly k' is a relative invariant of weight -1, and so one may take

$$k' = 1/k.$$

In four variables these tensors have been discussed by Eddington (1923, p. 107) under the name 'the alternating tensor of the 4th rank, and in 3 or 4 variables by Levi-Civita (1925, p. 159) as 'ϵ-systems', but the fundamental importance of these tensors does not appear to have been recognized.

It is now possible to obtain in specific form the results mentioned in the last section connecting the weight and rank of a tensor.

Let

$$A_{\alpha_1 \dots \alpha_m} = \phi B_{\alpha_1 \dots \alpha_m}$$

be a tensor of type $\{\lambda_1 + 1, \lambda_2 + 1, \dots, \lambda_n + 1\}$, where ϕ is the operator NP obtained from a Young tableau corresponding to this partition, in which for convenience we will suppose that the n symbols in the first column are the first n suffixes $\alpha_1, \dots, \alpha_n$. It is desirable to express $A_{\alpha_1 \dots \alpha_m}$ in this manner so as to ensure that it is expressed as a tensor in the correct form, but it is immaterial whether $B_{\alpha_1 \dots \alpha_m}$ is a tensor equivalent to $A_{\alpha_1 \dots \alpha_m}$, or a complete tensor of rank m, or some intermediate tensor, provided of course that the operation of ϕ does not give zero.

Then clearly N, and hence ϕ, can be expressed with a left-hand factor which is the negative symmetric group on the symbols $\alpha_1, \dots, \alpha_n$. Hence if S represents a positive and T a negative permutation of $\alpha_1, \dots, \alpha_n$, then

$$N = SN = -TN, \quad \phi = S\phi = -T\phi,$$

and

$$A_{\alpha_1 \dots \alpha_m} = SA_{\alpha_1 \dots \alpha_m} = -TA_{\alpha_1 \dots \alpha_m}.$$

Thus the $n!$ components of the tensor $A_{\alpha_1 \dots \alpha_m}$ obtained by permuting the first n suffixes are all equal, save for sign. Hence the effect of multiplying by the contragredient alternating tensor and contracting with respect to the first n suffixes is to add these equal terms, replacing them by a single component $n!$ times as large. Then from (1) and (2), p. 317, the tensor

$$C_{\alpha_{n+1} \dots \alpha_m} = E^{\alpha_1 \dots \alpha_n} A_{\alpha_1 \dots \alpha_n \alpha_{n+1} \dots \alpha_m}$$

is a tensor of type $\{\lambda_1, \dots, \lambda_n\}$. Owing to the weight of the alternating tensor, by this operation the weight of the tensor is increased by unity.

The tensor $A_{\alpha_1 \dots \alpha_m}$ can be obtained from $C_{\alpha_{n+1} \dots \alpha_m}$ by multiplication by the cogredient alternating tensor

$$n!\, A_{\alpha_1 \dots \alpha_m} = E_{\alpha_1 \dots \alpha_n} C_{\alpha_{n+1} \dots \alpha_m}.$$

For the generalization in which S-functions are allowed to have negative parts, the suffixes corresponding to these negative parts are left as uncontracted contragredient suffixes.

The alternating tensor with a metric and tensor densities

Sometimes tensors are used in conjunction with a fundamental quadratic form such as is used to define the *distance* between two points. This is called a *metric* form, and the tensor of coefficients is called the *metric tensor*, and is usually denoted by $g_{\mu\nu}$. The corresponding contragredient tensor $g^{\mu\nu}$ is defined so that

$$[g^{st}] = [g_{st}]^{-1},$$

and thus

$$g^{\lambda\mu} g_{\mu\nu} = \delta^{\lambda}_{\nu} \begin{cases} = 1 & \text{if } \lambda = \mu, \\ = 0 & \text{if } \lambda \neq \mu. \end{cases}$$

The determinant $|g_{st}|$ is denoted by g, and clearly $|g^{st}| = g^{-1}$.

If a metric is introduced it is convenient to take the arbitrary scalar k in the alternating tensor so that the suffixes can be raised or lowered by means of the metric tensor, i.e. so that

$$E_{i_1 \dots i_n} = g_{i_1 j_1} g_{i_2 j_2} \dots g_{i_n j_n} E^{j_1 \dots j_n}.$$

This will be the case if

$$k = gk^{-1},$$

so that $k = \sqrt{g}$. This value of k is always taken if there is a metric.

It may be noticed that in tensor calculus (Eddington 1923, p. 111), when a tensor is converted into a *tensor density*, a factor $1/\sqrt{g}$ is introduced, and a factor $\sqrt{g}$ is always introduced when a tensor is integrated through a volume. The necessity for the introduction of such an irrational expression as $\sqrt{g}$ would be otherwise difficult to explain, but it is clear that the presence of this factor is really indicative of the presence of the alternating tensor.

In fact, if the element of volume is a generalized parallelepiped with edges $\delta_1 x_i$, $\delta_2 x_i$, ..., $\delta_n x_i$, then the volume δV of this element is given by

$$\delta V = E^{i_1 \dots i_n} \delta_1 x_{i_1}\, \delta_2 x_{i_2} \dots \delta_n x_{i_n},$$

which shows the appropriate position of the alternating tensor in an integration formula.

It is a characteristic property of the alternating tensor when taken over the orthogonal group, i.e. the group of transformations which leave the metric tensor invariant, that it is invariant for a proper rotation, but is changed in sign by an improper rotation, i.e. an orthogonal transformation of negative determinant. This property is shared by any expression giving the volume of a region, and this confirms that the integration over a volume should properly involve the alternating tensor.*

The δ-symbols

Certain combinations of alternating tensors are of interest.

Clearly
$$E^{i_1 \dots i_n} E_{i_1 \dots i_n} = n!.$$

Also
$$E^{ij_2 \dots j_n} E_{kj_2 \dots j_n} \begin{cases} = (n-1)! & \text{if} \quad i = k, \\ = 0 & \text{if} \quad i \neq k. \end{cases}$$

Putting $\delta^i_i = 1$, $\delta^i_k = 0$, $i \neq k$, this tensor may be expressed as $(n-1)!\,\delta^i_k$.

Again
$$E^{i_1 i_2 j_3 \dots j_n} E_{k_1 k_2 j_3 \dots j_n} = (n-2)!\,(\delta^{i_1}_{k_1}\delta^{i_2}_{k_2} - \delta^{i_1}_{k_2}\delta^{i_2}_{k_1})$$
$$= (n-2)!\,\delta^{i_1 i_2}_{k_1 k_2}.$$

This symbol $\delta^{i_1 i_2}_{k_1 k_2}$ is used to denote $+1$ if the lower suffixes form a positive permutation of the upper suffixes, -1 if a negative permutation, and 0 if either the upper or the lower suffixes have a repeated symbol or if the lower suffixes do not represent a permutation of the upper suffixes.

The definition of $\delta^{i_1 \dots i_r}_{k_1 \dots k_r}$ is similar, and clearly
$$E^{i_1 \dots i_r j_{r+1} \dots j_n} E_{k_1 \dots k_r j_{r+1} \dots j_n} = (n-r)!\,\delta^{i_1 \dots i_r}_{k_1 \dots k_r}.$$

From the manner of their formation from alternating tensors, these symbols are clearly tensors.

Simple variable tensors

In this section, since contragredient variables are being dealt with, when reference is made to a tensor of type $\{\lambda\}$, a tensor of *contragredient* type $\{\lambda\}$ is to be understood.

The product of r identical tensor variables, e.g.
$$x^{i_1} x^{i_2} \dots x^{i_r},$$
clearly forms a symmetric tensor, or a simple tensor of type $\{r\}$. To obtain a tensor corresponding to a partition into more than one part, more than one set of variables is necessary. Products of variables from different sets, however, do not form a *simple*, but a *complex* tensor, as a simple example will show. Thus
$$x_i y_j = \tfrac{1}{2}(x_i y_j + x_j y_i) + \tfrac{1}{2}(x_i y_j - x_j y_i),$$
and the product $x_i y_j$ is clearly equivalent to the sum of a symmetric and an antisymmetric tensor.

To separate products of variables into simple tensors the δ-symbols are used. The tensor
$$x^{i_1 \dots i_r} = \delta^{i_1 \dots i_r}_{j_1 \dots j_r} x^{j_1} y^{j_2} \dots w^{j_r}$$
is clearly an antisymmetric tensor, or a simple tensor of type $\{1^r\}$. As a set of variables, this tensor is usually called the rth compound of the original variables $x, y, \dots$. Assume now that the sets of variables are ordered in some way, and that $x^i, y^i, \dots, w^i$ represent the first r sets of variables. Then $x^{i_1 \dots i_r}$ will henceforward denote this tensor.

* See Eddington (1936, p. 58); iE_5 corresponds to the alternating tensor.

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Next it is shown that a simple tensor of type $\{\lambda_1, \ldots, \lambda_n\}$ can be obtained by multiplying tensors of this form.

LEMMA I. *A simple tensor of type $\{\lambda_1, \lambda_2, \ldots, \lambda_p\}$ obtained from p sets of variables $x^i_{(1)}, x^i_{(2)}, \ldots, x^i_{(p)}$ is of degree at least λ_p in the variables $x^i_{(p)}$.*

The number of variables in each set is taken to be p, putting $x^{p+1}_{(i)} = x^{p+2}_{(i)} = \ldots = 0$ if necessary. Since the number of parts in the S-function $\{\lambda\}$ does not exceed the number of variables, the S-function will not be identically zero, and neither will the tensor of type $\{\lambda\}$.

If, however, a linear relation

$$\Sigma a_j x^j_{(i)} \quad (1 \leqslant j \leqslant p)$$

is introduced in each set of variables, the number of linearly independent variables is reduced to $(p-1)$. Hence since the number of parts now exceeds the number of variables, the tensor must be identically zero.

Thus, by the remainder theorem, the tensor is exactly divisible by $|x^t_{(s)}|$. Since $|x^t_{(s)}|$ is a relative invariant, being a tensor of type $\{1^p\}$, the quotient is clearly a tensor of type $\{\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_p - 1\}$.

Repetition of the argument shows that the tensor is divisible by $|x^t_{(s)}|^{\lambda_p}$, and the quotient is then a tensor of type $\{\lambda_1 - \lambda_p, \lambda_2 - \lambda_p, \ldots, 0\}$. The lemma follows.

COROLLARY. If the degree of the tensor of type $\{\lambda\}$ in the variables $x^i_{(p)}$ is exactly λ_p, then the degree in the variables $x^i_{(p-1)}$ is $\geqslant \lambda_{p-1}$. If the degree in the variables $x^i_{(p)}$ is λ_p and in the variables $x^i_{(p-1)}$ is λ_{p-1}, then the degree in the variables $x^i_{(p-2)}$ is $\geqslant \lambda_{p-2}$, and so on.

LEMMA II. *The product of two tensors of types $\{\lambda\}$ and $\{\mu\}$ respectively is expressible as a sum of tensors of those types which correspond to the S-functions appearing in the product $\{\lambda\}\{\mu\}$.*

To prove this it is observed that the matrix of transformation of the product of the tensors is the direct product of the matrices of transformation, and thus the spur is the product of the spurs, namely, $\{\lambda\}\{\mu\}$. It does not follow that for each S-function in the product $\{\lambda\}\{\mu\}$ there will be a non-zero tensor, for the corresponding terms may be identically zero, e.g. in the product tensor $x^i x^j$, the terms $x^i x^j - x^j x^i$ are identically zero, and the tensor of type $\{1^2\}$ does not arise.

The essential theorem concerning variable tensors may now be proved.

THEOREM. *The product $(x^i)^{\lambda_1-\lambda_2}(x^{ij})^{\lambda_2-\lambda_3}(x^{ijk})^{\lambda_3-\lambda_4}\ldots$ is a simple tensor of type $\{\lambda_1, \lambda_2, \ldots, \lambda_p\}$.*

By $(x^i)^{\lambda_1-\lambda_2}$ is meant the product of $(\lambda_1 - \lambda_2)$ tensors each equal to x^i. There will be $(\lambda_1 - \lambda_2)$ different suffixes. Similarly for $(x^{ij})^{\lambda_2-\lambda_3}$, which will have $2(\lambda_2 - \lambda_3)$ different suffixes.

The tensors which appear in the product will be of rank $(\lambda_1 + \lambda_2 + \ldots + \lambda_p)$. Since x^i is of type $\{1\}$, x^{ij} of type $\{1^2\}$, etc., then if a tensor of type $\{\mu_1, \mu_2, \ldots, \mu_n\}$ appears in the product, the S-function $\{\mu_1, \mu_2, \ldots, \mu_n\}$ must appear in the product

$$\{1\}^{\lambda_1-\lambda_2}\{1^2\}^{\lambda_2-\lambda_3}\{1^3\}^{\lambda_3-\lambda_4}\ldots.$$

It will follow that
$$\lambda_1 + \lambda_2 + \ldots + \lambda_p = \mu_1 + \mu_2 + \ldots + \mu_n,$$

and also
$$\mu_1 \leqslant \lambda_1, \; \mu_1 + \mu_2 \leqslant \lambda_1 + \lambda_2, \; \mu_1 + \mu_2 + \mu_3 \leqslant \lambda_1 + \lambda_2 + \lambda_3, \; \ldots.$$

Further, since the number of sets of variables is p, the degree of the tensor in the variables $x_{(p)}$ is λ_p, and in the variables $x_{(p-1)}$ is λ_{n-1}, etc., it follows that

$$\mu_{p+1} = 0, \quad \mu_p \leqslant \lambda_p, \quad \mu_{p-1} \leqslant \lambda_{p-1}, \quad \ldots,$$

and therefore $\mu_1 = \lambda_1, \quad \mu_2 = \lambda_2, \quad \ldots, \quad \mu_p = \lambda_p,$

and $\{\mu\} = \{\lambda\}.$

Since, further, the coefficient of $\{\lambda\}$ in the product $\{1\}^{\lambda_1-\lambda_2}\{1^2\}^{\lambda_2-\lambda_3}\{1^3\}^{\lambda_3-\lambda_4}\ldots$ is unity, it must follow that the tensor is simple.

The tensor defined in the theorem is called a *Clebsch variable tensor*, after Clebsch (1872), who stated precisely the fundamental character of these sets of variables.

The fundamental property of these tensors, which follows immediately from the definition, is as follows:

THEOREM. *The product of two simple Clebsch tensors of type* $\{\lambda_1, \lambda_2, \ldots, \lambda_p\}$ *and* $\{\mu_1, \mu_2, \ldots, \mu_p\}$ *respectively is a simple Clebsch tensor of type* $\{\lambda_1+\mu_1, \lambda_2+\mu_2, \ldots, \lambda_p+\mu_p\}$.

COGREDIENT VARIABLES

Sometimes, in addition to the various sets of contragredient variables x^i, y^i, z^i, ..., some sets of cogredient variables u_i, v_i, w_i, ... are used. These may be treated in the same manner as contragredient variables, and the corresponding tensors may be converted into contragredient tensors by the use of the alternating tensor. An example in four variables will suffice.

Let x^i, y^i, z^i, ξ^i be four sets of contragredient variables. Now choose the cogredient variables so that

$$u_i = E_{ijkm} x^j y^k z^m, \qquad v_i = E_{ijkm} x^j y^k \xi^m,$$
$$w_i = E_{ijkm} x^j z^k \xi^m, \qquad \omega_i = E_{ijkm} y^j z^k \xi^m.$$

Hence, since $E_{ijkm} x^i y^j z^k \xi^m = \Delta$, an invariant,

it follows that $u_i \xi^i = v_i z^i = w_i y^i = \omega_i x^i = \Delta,$

$$u_i x^i = u_i y^i = u_i z^i = v_i x^i = v_i y^i = v_i \xi^i = w_i x^i$$
$$= w_i z^i = w_i \xi^i = \omega_i y^i = \omega_i z^i = \omega_i \xi^i = 0.$$

From the properties of determinants it follows that

$$u_{ij} = \delta_{ij}^{i'j'} u_{i'} v_{j'} = \Delta E_{ijkm} x^{km}, \quad u_{ijk} = \delta_{ijk}^{i'j'k'} u_{i'} v_{j'} w_{k'} = \Delta^2 E_{ijkm} x^m.$$

Thus the cogredient variable tensors u_i, u_{ij}, u_{ijk} are equivalent to the contragredient tensors x^{ijk}, x^{ij}, x^i.

ALGEBRAIC FORMS

An algebraic form $f(a_i; x^k_{(j)})$ or $f(a_i; x^k_{(j)}; u^{(r)}_m)$ is called an *invariant* form if it is a polynomial in any number of sets of contragredient variables $x^k_{(j)}$, and also possibly sets of cogredient variables $u^{(r)}_m$, and if, whenever the variables are subjected to a linear transformation, the coefficients a_i are also subjected to a transformation in such a manner that the form f remains invariant, absolute or relative.

The set of coefficients clearly form a tensor. The form itself may be regarded as a tensor of zero rank. If the tensor of coefficients is *simple* or *complex*, the corresponding form is said to be *simple* or *complex*. A complex form may clearly be expressed as a sum of simple forms.

Two simple algebraic forms which have the same tensor of coefficients are said to be *equivalent*. They may be obtained from one another by operators of the type $y_i \frac{\partial}{\partial x_i}$. These are called polar operators and the process is called polarization (Turnbull 1928).

A fundamental problem of invariant theory

In the general problem of invariant theory some set of invariant forms called *ground forms* are given, each of which may be supposed to be simple, each of these having an arbitrary independent coefficient for each term of the corresponding tensor of variables. Then in addition to these forms many other invariant forms may be written down whose coefficients are polynomials in the coefficients of the ground forms. These are called *concomitants*. It is a major problem of invariant theory to determine the complete set of concomitants of a given set of ground forms.

Reducibility

The set of concomitants is obviously infinite in number, and further, the number of linearly independent concomitants is infinite, for the powers and products of concomitants are clearly concomitants. To reduce this infinite set to a finite set, the concept of *reducibility* is introduced. Hilbert (1890) has shown that for any set of ground forms there exists a finite set of concomitants, called a *basis*, such that every concomitant may be expressed as a polynomial in the concomitants of the basis with fixed coefficients.

For a given set of ground forms such a basis may be constructed by examining the concomitants in ascending degree in the coefficients of the ground forms, one by one. Those concomitants which can be expressed as polynomials in the concomitants of lower degree are said to be *reducible* and are rejected. A concomitant which is not reducible is *irreducible* and is retained as a member of the basis.

Concomitants which are *equivalent* to a given concomitant, i.e. having the same tensor of coefficients, are not treated as separate concomitants.

A certain complication arises at this point, however, because, if X and Y are two invariant forms, and X' and Y' are two other forms equivalent to them, then $X'Y'$ is not necessarily equivalent to XY. A simple example will illustrate this.

The product of the two simple linear forms $a_i x^i$, $b_i x^i$ is the simple quadratic form $a_i b_j x^i x^j$. But the product of the two forms $a_i x^i$, $b_i y^i$ is a complex form

$$a_i b_j x^i y^j = \tfrac{1}{2}(a_i b_j + a_j b_i)\,(x^i y^j + x^j y^i) + \tfrac{1}{2}(a_i b_j - a_j b_i)\,(x^i y^j - x^j y^i).$$

The whole concept of reducibility is in this way imperilled. Thus the quadratic $a_{ij} x^i x^j$ has a concomitant

$$\begin{vmatrix} a_{ij} & a_{im} \\ a_{kj} & a_{km} \end{vmatrix} \cdot \begin{vmatrix} x_i & x_k \\ y_i & y_k \end{vmatrix} \cdot \begin{vmatrix} x_j & x_m \\ y_j & y_m \end{vmatrix},$$

which is usually regarded as irreducible, but which can be expressed as a polynomial in the ground form and equivalent forms, namely,

$$(a_{ij} x^i x^j)\,(a_{km} y^k y^m) - (a_{ij} x^i y^j)^2.$$

In a similar manner every concomitant of a given set of ground forms may be expressed as a polynomial in these ground forms and equivalent forms.

Such a basis, however, would be of no practical significance, as it would certainly not convey all the information regarding the concomitants.

The situation is restored by the restriction that all ground forms and all concomitants of the basis shall be expressed as *Clebsch forms*. For any given algebraic form there is one and only one equivalent Clebsch form. A concomitant is said to be reducible only if the equivalent Clebsch form is expressible as a polynomial in the Clebsch forms of the basis *without polarization*.

A Clebsch form is a form in which the tensor of variables is a Clebsch tensor as previously defined. These have the fundamental property that the product of two simple Clebsch forms is a simple Clebsch form, a fact which follows immediately from the corresponding property of Clebsch variable tensors.

With this restriction the basis of a given set of ground forms is obviously unique, save that to each irreducible concomitant may be added any polynomial in the irreducible concomitants of lower degree provided that the degree in the coefficients of each ground form, and also the type, i.e. the partition with which it is associated, is the same as for the irreducible concomitant.

If the *basis* of a given set of ground forms is known, and also the *syzygies*, i.e. the algebraic relations which are satisfied by the concomitants of the basis, then a complete knowledge of all the concomitants follows.

Construction of concomitants

A comparatively simple method for the construction of concomitants, but one which, though persistently used in Riemannian geometry and in physics, has not been generally employed in invariant theory, is by the multiplication and contraction of tensors. Tensor coefficients of the ground forms are multiplied together in any manner. The product is then multiplied by the alternating tensor if and as often as desired, and by a Clebsch variable tensor in such a manner that the number of upper and lower suffixes are equal. If then the process of contraction is used to eliminate all suffixes a tensor of rank zero is obtained, and therefore, if the result is not identically zero, a concomitant of the ground forms.

By this means an unlimited number of concomitants may be written down. The drawback is that many such expressions on evaluation prove to be identically equal to zero, many different expressions represent the same concomitant, and many prove to be linearly dependent. A technique is needed to distinguish the independent concomitants. However, it is necessary first to show that all concomitants can be obtained by this method.

The fundamental theorem

Every concomitant of a given set of ground forms may be obtained by multiplying tensor coefficients, tensor variables, the alternating tensor as required, and contracting.

Let the variables undergo the transformation

$$x'^j = \xi^j_i x^i, \quad x^i = \eta^i_j x'^j.$$

Every concomitant may be expressed as a sum of concomitants each of which is a simple form and is homogeneous in the coefficients of each ground form. It is assumed that only one such simple form is considered.

Let the concomitant be expressed in the form

$$f = P^{i_1 \dots i_q}_{j_1 \dots j_r} a_{\alpha_1 \alpha_2 \dots} b_{\beta_1 \beta_2 \dots} \dots x^{\gamma_1 \gamma_2 \dots} \dots,$$

where $a_{\alpha_1 \alpha_2 \dots}$, $b_{\beta_1 \beta_2 \dots}$, etc. are tensor coefficients, $x^{\gamma_1 \gamma_2 \dots}$, etc. are tensor variables and P is a numerical coefficient which is given an upper suffix equal to each lower suffix in the accompanying tensors, and a lower suffix equal to each upper suffix.

These coefficients P may be chosen so that they satisfy the same symmetrizing relations in respect of their suffixes as the tensors which they accompany, e.g. if one of the tensors a_{ij} is symmetric or antisymmetric in these two suffixes, then coefficient P also will be symmetric or antisymmetric in these suffixes. Further, if the concomitant is of degree >1 in any of the ground forms, it can be arranged so that the coefficient P will be unaltered for the interchange of any two tensors obtained from the same ground form. This can be assured if P is operated on with all the idempotent operators, permuting the suffixes in $a_{\alpha_1 \alpha_2 \dots} b_{\beta_1 \beta_2 \dots} x^{\gamma_1 \gamma_2 \dots}$, which leave this tensor product unaltered.

If this be done, then, if the concomitant is given the coefficients P will be uniquely determined, and further, after a transformation the coefficients will still possess the same properties.

The coefficients P are thus invariants of the transformations, but because of the invariance of the form f, the coefficients P will also transform like a tensor.

Thus after transformation, an identical relation connecting the elements ξ^i_j and η^i_j of the transforming matrix and its reciprocal, arises, i.e.

$$P^{i_1 i_2 \dots i_q}_{j_1 j_2 \dots j_r} = P'^{i_1 \dots i_q}_{j_1 \dots j_r} = P^{i_1' \dots i_q'}_{j_1' \dots j_r'} \xi^{i_1}_{i_1'} \xi^{i_2}_{i_2'} \dots \xi^{i_q}_{i_q'} \eta^{j_1'}_{j_1} \dots \eta^{j_r'}_{j_r}.$$

If the weight of the concomitant f is not zero, allowance must be made for a power of the determinant Δ. To avoid any such modification the group of transformations is restricted so that the determinant $|\xi^s_t|$ is unity. The matrix $[\xi_{st}]$ is otherwise in no way restricted, the only algebraic relation between its n^2 components is that $|\xi^s_t| = 1$. This relation may be expressed

$$E^{i_1 i_2 \dots i_n} \xi^{i_1'}_{i_1} \xi^{i_2'}_{i_2} \dots \xi^{i_n'}_{i_n} = E^{i_1' \dots i_n'}.$$

Again, the elements of the reciprocal matrix $[\eta^s_t]$ may be expressed in terms of the ξ^i_j by means of the alternating tensor, e.g.,

$$(n-1)!\, \eta^i_j = E^{i i_2 \dots i_n} E_{j j_2 \dots j_n} \xi^{j_2}_{i_2} \xi^{j_3}_{i_3} \dots \xi^{j_n}_{i_n}.$$

Hence every algebraic relation connecting the quantities ξ^i_j, η^i_j can be expressed by combinations of alternating tensors. This holds in particular for the algebraic relation obtained above. It follows that the tensor of coefficients $P^{i_1 \dots i_r}_{j_1 \dots j_q}$ can be expressed as a combination of alternating tensors, and the theorem follows.

The symbolic method

It has been stated that the method of tensors has not generally been used in invariant theory. A powerful method which is in general use, and which, in its elementary application, is more or less equivalent to the method of tensors, is the symbolic method (see Turnbull

1928). A complete account of this here would be superfluous, but since its relation with the tensor method is to be shown, it will be illustrated with reference to the binary cubic.

The binary cubic $$a_0x+3a_1x^2y+3a_2xy^2+a_3y^3$$

is represented symbolically by the form

$$(\alpha_1x+\alpha_2y)^3,$$

so that the products α_1^3, $\alpha_1^2\alpha_2$, $\alpha_1\alpha_2^2$, α_2^3 are interpreted respectively as a_0, a_1, a_2, a_3. It is true that the symbolic expressions satisfy certain algebraic identities which are not satisfied in general by the coefficients, e.g. $a_0a_2=a_1^2$, $a_0a_3=a_1a_2$, etc., but such errors as would arise from these identities are avoided by the restricting condition that every expression shall be linear in the coefficients and hence of degree exactly 3 in the symbols. If it is desired to write down expression of degree greater than unity in the coefficients, a different system of symbols is employed, and the cubic is represented alternatively as $(\beta_1x+\beta_2y)^3$, $(\gamma_1x+\gamma_2y)^3$, etc., and such expressions only are employed which are of degree exactly 3 in each set of symbols.

The two expressions $(\alpha_1x+\alpha_2y)$ and $(\alpha_1\beta_2-\alpha_2\beta_1)$ remain invariant under a transformation as do such other expressions as might be obtained from these by changing the symbols for other sets. Products of expressions of these forms are taken, with the restriction that the degree in each set of symbols is exactly 3. The expressions are multiplied out and each term interpreted in terms of the coefficients in the cubic, a_0, a_1, a_2, a_3. Because of the invariance of the factor forms $(\alpha_1x+\alpha_2y)$ and $(\alpha_1\beta_2-\alpha_2\beta_1)$, the final expression also will be invariant, and thus gives a concomitant of the cubic.

A fundamental theorem is proved that every concomitant can be expressed symbolically in terms of these two fundamental factors.

The connexion with the method of tensors is quite simple. Each set of three symbols α, β, or γ corresponds to a tensor of coefficients of the cubic. Each factor $(\alpha_1x+\alpha_2y)$ indicates that one of the lower suffixes of the tensor is contracted with an upper suffix of a variable x^i. Each factor $(\alpha_1\beta_2-\alpha_2\beta_1)$ indicates that the two suffixes, one from each of the corresponding tensors, are contracted with those of an alternating tensor E^{ij}.

As an example, the binary cubic has a quadratic covariant of degree 2 in the coefficients which explicitly is

$$2[(a_0a_2-a_1^2)\,x^2+(a_0a_3-a_1a_2)\,xy+(a_1a_3-a_2^2)\,y^2].$$

The symbolic expression for this is

$$(\alpha_1\beta_2-\alpha_2\beta_1)^2\,(\alpha_1x+\alpha_2y)\,(\beta_1x+\beta_2y),$$

and the expression in tensors is

$$A_{ijk}A_{lmn}E^{il}E^{jm}x^kx^n.$$

This correspondence between the symbolic and tensorial expression of a concomitant becomes apparent in the general case if both methods are used simultaneously, the tensors being factorized symbolically as products of tensors of rank unity.

The symbolic method is justified by the far-reaching results that can be obtained by its use, but the artifice of representing the general cubic by one which is an exact cube involves an apparent deviation from fact which is distasteful to some people. I believe that it has not

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previously been recognized that this objection can very easily be overcome, and the symbolic method placed on a sound logical basis as follows (see also Turnbull 1928; Grace & Young 1903, pp. 365–366).

Although every binary cubic is not an exact cube, it is expressible as a sum of cubes. The number of cubes which must appear in the sum is irrelevant. The cubic can thus be expressed as

$$\Sigma\alpha_1^3 x^3 + 3\Sigma\alpha_1^2\alpha_2 x^2 y + 3\Sigma\alpha_1\alpha_2^2 xy^2 + \Sigma\alpha_2^3 y^3,$$

the summation being with respect to the various cubes involved.

Every symbolic expression which is of degree exactly 3 in the α's may thus be summed with respect to these cubes and the symbolic products will be replaced by the corresponding coefficients.

In the general case, for a ground form of type $\{\lambda\} \equiv \{\lambda_1, \lambda_2, \ldots, \lambda_p\}$ in any number of variables, the tensor of coefficients is replaced by a tensor constructed by means of symbols in *exactly the same manner* as the Clebsch coefficient tensor is constructed from the variables, i.e. the symbolic form is expressed as (Clebsch 1872; Capelli 1880, 1882; Clebsch & Capelli 1891, 1892; Clebsch & Gordan 1872, 1875)

$$(\alpha_i x^i)^{\lambda_1-\lambda_2}\{(\alpha_i\beta_j-\alpha_j\beta_i)(x^iy^j-x^jy^i)\}^{\lambda_2-\lambda_3}\{(\alpha_i\beta_j\gamma_k-\ldots)(x^iy^jz^k-\ldots)\}^{\lambda_3-\lambda_4}\ldots.$$

To justify the use of such a form it is necessary to show that every simple form of this type may be expressed as a sum of symbolic forms of this type. This is proved by a *reductio ad absurdum*. The set of forms which *can* be expressed as a sum of symbolic forms of this kind is clearly a linear set which is closed to the group of linear transformations. If this does not include every form of type $\{\lambda\}$, then the set of forms of type $\{\lambda\}$ possesses a subset which is closed to the group of transformations. Hence the forms of type $\{\lambda\}$ are not simple but complex, contrary to hypothesis. Thus every form of type $\{\lambda\}$ can be expressed as a sum of symbolic forms, and the use of the symbolic method is justified.

The real problem of invariant theory consists in the selection of the irreducible concomitants from the large body of forms given by the symbolic or tensor method. A considerable body of technique has been produced in connexion with symbolic method to this end. Whether this technique could be translated effectively into the tensor notation cannot be said, as the attempt has not been made. But the method of tensors seems to lend itself more effectively to another approach based on group characters, group representational theory and the quantitative substitutional methods of A. Young. This will be dealt with in Part II of this paper.

Concomitants under restricted groups

The group of transformations so far considered has been taken either as the full linear group, or as the group of transformations with unit determinant. The transformations may, however, be restricted to some special kinds of transformation, e.g. the orthogonal group or the simplectic group.

Such groups will be subgroups of the full linear group. The representations of the group may be obtained from the representations of the full linear group.

An irreducible representation of the full linear group will of necessity give a representation of the subgroup, but such a representation may be reducible. The reduction of these representations into irreducible components will give all the representations of the restricted group. There will, however, be repetitions, the same representation of the restricted group appearing as component parts of different representations of the full linear group.

Corresponding to each decomposition of a simple representation of the full linear group into irreducible components we have a similar decomposition of the corresponding tensors.

Thus a tensor which is simple over the full linear group may be complex over, for example, the orthogonal group. And the decomposition into simple tensors follows exactly the decomposition of the simple representation of the full linear group into simple representations of the restricted group.

An essential preliminary to the study of concomitants over a restricted group is an examination of the simple representations of the group, for only thus can the simple algebraic forms over the group be determined.

The restricted group is generally taken to be the group which leaves a certain form or forms invariant (see Klein 1921, pp. 409 et seq.). Thus the orthogonal group leaves a quadratic form invariant, the simplectic group leaves a linear complex invariant. Thus there is available, as well as the alternating tensor, the tensors of coefficients of these invariant forms which will be called *fundamental tensors*.

Considerations of space prevent a full account here, but it is hoped to develop the corresponding theory for restricted groups in another paper.

PART II. GROUP REPRESENTATIONAL METHODS

The new multiplication of S-functions

In a previous note (Littlewood 1936 *b*) an application of group characters and S-functions to invariant theory was described. A brief resume is given here and further generalizations obtained. The methods of the note gave only the number and type of the concomitants. The corresponding results in representational theory and substitutional analysis also are developed here, by means of which the actual concomitants can be constructed. Further extensions of the theory are made to concomitants over the orthogonal group.

Let $f(x^i) \equiv f(x^1, x^2, \ldots, x^n)$ be an n-ary p-ic, i.e. a polynomial of the pth degree in n variables x^i. If X denotes the column vector $[x^s]$ and $X^{\{p\}}$ denotes the pth induced matrix of X, then the polynomial f may be expressed in the form

$$f = FX^{\{p\}},$$

where F is the row vector whose elements are the coefficients in f.

When the variables undergo a linear transformation, this may be expressed in the form, remembering that the variables are *contragredient*,

$$X' = A^{-1}X, \quad \text{or} \quad X = AX',$$

A being the matrix of the transformation. Then $X^{\{p\}}$ will be transformed by the pth induced matrix of A, i.e.

$$X^{\{p\}} = A^{\{p\}}X'^{\{p\}}.$$

Hence

$$f = F'X'^{\{p\}} = FX^{\{p\}} = FA^{\{p\}}X'^{\{p\}},$$

and thus

$$F' = FA^{\{p\}}.$$

The manner of transformation of the homogeneous products of degree q of the coefficients forming F is sought. This will be given by the equation of transformation of the qth induced matrix of F, i.e.

$$F'^{\{q\}} = F^{\{q\}}\,[A^{\{p\}}]^{\{q\}}.$$

The matrix of transformation is thus the qth induced matrix of the pth induced matrix of A.

According to Schur's definition of an invariant matrix, an invariant matrix of an invariant matrix is clearly an invariant matrix, though not in general irreducible. An induced matrix of an induced matrix will thus be an invariant matrix, but in general reducible, and equivalent to a direct sum of irreducible invariant matrices in the form

$$[A^{\{p\}}]^{\{q\}} = \dot{\Sigma} A^{\{\lambda\}},$$

the symbol $\dot{\Sigma}$ denoting direct sum, and $A^{\{\lambda\}}$ denoting the invariant matrix of A corresponding to the partition $(\lambda) \equiv (\lambda_1, \lambda_2, \ldots, \lambda_i)$.

To each such term $A^{\{\lambda\}}$ there corresponds a set of linear combinations of products of degree q of the coefficients in F, such that these transform in the manner of a tensor of type $\{\lambda\}$. Multiplying by the appropriate Clebsch variable tensor and contracting, a concomitant of type $\{\lambda\}$ is obtained.

Now define a *new multiplication of S-functions* represented by the symbol $\otimes$, such that, if (λ), (μ), (ν) represent partitions, and

$$[A^{\{\lambda\}}]^{\{\mu\}} = \dot{\Sigma} A^{\{\nu\}},$$

then

$$\{\lambda\} \otimes \{\mu\} = \Sigma\{\nu\}.$$

The following theorem is thus demonstrated:

THEOREM. *If*

$$\{p\} \otimes \{q\} = \Sigma\{\lambda\},$$

then for each S-function $\{\lambda\}$ which appears in the sum the n-ary p-ic has a concomitant of degree q in the coefficients, and of type $\{\lambda\}$.

The generalization to a ground form of type $\{\mu\}$ is obvious. The S-function $\{p\}$ which has hitherto been restricted to correspond to a partition into one part only is replaced by the general S-function $\{\mu\}$, and the equation becomes

$$\{\mu\} \otimes \{q\} = \Sigma\{\lambda\}.$$

It may be noticed that in this application to invariants the right-hand factor in a *new multiplication* is usually a partition into one part only, while the left factor may represent any S-function.

For a set of two ground forms corresponding to partitions $\{\mu\}$ and $\{\nu\}$, the concomitants which are of degree q in the first and r in the second set of coefficients are sought.

The matrix of transformation of homogeneous products of degree q in the coefficients of the first corresponds to the expression

$$\{\mu\} \otimes \{q\},$$

and the matrix of transformation of homogeneous products of degree r in the second corresponds to

$$\{\nu\} \otimes \{r\}.$$

The matrix of transformation of homogeneous products which are of degree q in the first and r in the second will clearly be the direct product of these two matrices, and since the spur of a direct product is the product of the spurs will correspond to

$$[\{\mu\}\otimes\{q\}]\,[\{\nu\}\otimes\{r\}],$$

the multiplication between the brackets being the ordinary multiplication of S-functions.

Hence *if* $$[\{\mu\}\otimes\{q\}]\,[\{\nu\}\otimes\{r\}] = \Sigma\{\lambda\}$$

there will be a simultaneous concomitant of the two ground forms of degree q in the first and r in the second, and of type $\{\lambda\}$ for each S-function $\{\lambda\}$ in the summation.

A similar result holds for any number of ground forms.

A particularly simple case is the problem of finding the concomitants which are linear in each of a given set of ground forms. The new multiplication is not then required. If the types of the ground forms are respectively $\{\mu\}$, $\{\nu\}$, $\{\xi\}$, ..., $\{\zeta\}$, then the types of concomitant are given by

$$\{\mu\}\{\nu\}\{\xi\}\ldots\{\zeta\} = \Sigma\{\lambda\}.$$

The ordinary multiplication of S-functions is easily evaluated (Littlewood & Richardson 1934*b*; Littlewood 1940, p. 94). The determination of $\{\mu\}\otimes\{\nu\}$ is not so easy and requires a technique. This will be considered in Part III of this paper.

Some properties of the new multiplication are important, and five theorems concerning the operation $\otimes$ are now proved.

First, $A^{\{\mu\}\dot{+}\{\nu\}}$ is taken to mean $A^{\{\mu\}}\dot{+}A^{\{\nu\}}$, $\dot{+}$ denoting direct sum. It follows immediately that:

(I) *The operation $\otimes$ is distributed with respect to addition on the right only.* Concerning addition on the left, it is noted that if $\{\lambda\}'$ and $\{\lambda\}''$ denote S-functions of different sets of p and q quantities respectively, and $\{\lambda\}$ denotes an S-function of the complete set of $(p+q)$ quantities, then (Littlewood 1940, p. 105)

$$\{\lambda\} = \Sigma g_{\mu\nu\lambda}\{\mu\}'\{\nu\}'',$$

the summation being with respect to all pairs of S-functions $\{\mu\}$, $\{\nu\}$, such that $\{\lambda\}$ appears in the product $\{\mu\}\{\nu\}$ with coefficient $g_{\mu\nu\lambda}$, including the cases $\{\mu\}=\{0\}=1$, $\{\nu\}=\{\lambda\}$ and $\{\mu\}=\{\lambda\}$, $\{\nu\}=\{0\}=1$.

It follows that

(II) $$[\{\mu\}+\{\nu\}]\otimes\{\lambda\} = \Sigma g_{\alpha\beta\lambda}[\{\mu\}\otimes\{\alpha\}]\,[\{\nu\}\otimes\{\beta\}].$$

In the above (Littlewood 1940, p. 99), if $\{\lambda\}'$, $\{\lambda\}''$ and $\{\lambda\}$ are associated respectively with the series f, g and $F=fg$, then if h represents the series g^{-1}, it follows that $f=Fh$. Denote S-functions associated with h by $\{\lambda\}'''$. Then if (λ) is a partition of d

$$\{\lambda\}''' = (-1)^d\{\tilde{\lambda}\}'',$$

$(\tilde{\lambda})$ denoting the partition conjugate to (λ).

Hence $$\{\lambda\}' = \Sigma g_{\mu\nu\lambda}\{\mu\}\{\nu\}'''$$
$$= \Sigma(-1)^d\,g_{\mu\nu\lambda}\,\{\mu\}\{\tilde{\nu}\}''.$$

It follows that

(III) $$[\{\mu\}-\{\nu\}]\otimes\{\lambda\} = \Sigma\,(-1)^d g_{\alpha\beta\lambda}\,[\{\mu\}\otimes\{\alpha\}]\,[\{\nu\}\otimes\{\tilde{\beta}\}],$$

(β) *being a partition of d.*

(IV) *The operation* $\otimes$ *is associative.*

Let $K = [[A^{\{\lambda\}}]^{\{\mu\}}]^{\{\nu\}}$ denote the invariant matrix corresponding to $\{\nu\}$ of the invariant matrix corresponding to $\{\mu\}$ of the invariant matrix of A corresponding to the partition $\{\lambda\}$. If that portion $[A^{\{\lambda\}}]^{\{\mu\}}$ of the expression for K is replaced by the equivalent direct sum of invariant matrices, the effect is merely to transform it by a fixed matrix, and hence K also is transformed by a fixed matrix, its canonical form remaining unaltered. Similarly, if $B = A^{\{\lambda\}}$ and $[B^{\{\mu\}}]^{\{\nu\}}$ is replaced by the equivalent sum of invariant matrices, then the canonical form of K is unaltered. Taking the spur of K it follows that

$$[\{\lambda\}\otimes\{\mu\}]\otimes\{\nu\} = \{\lambda\}\otimes[\{\mu\}\otimes\{\nu\}],$$

and the theorem is proved.

Lastly, consider $[\{\lambda\}\{\mu\}]\otimes\{\nu\}$. If (α) is a partition of n, let $\chi_\rho^{(\alpha)}$ denote the characteristic of the class ρ of the symmetric group of order $n!$ corresponding to the partition (λ). Then since the product of two simple characters of the group is also a character, simple or compound if (β) also is a partition of n, $\chi_\rho^{(\alpha)}\chi_\rho^{(\beta)}$ may be expressed in the form

$$\chi_\rho^{(\alpha)}\chi_\rho^{(\beta)} = \Sigma K_{\alpha\beta\nu}\chi_\rho^{(\nu)}.$$

With this definition of $K_{\alpha\beta\nu}$ the theorem is as follows:

(V) $$[\{\lambda\}\{\mu\}]\otimes\{\nu\} = \Sigma K_{\alpha\beta\nu}[\{\lambda\}\otimes\{\alpha\}]\,[\{\mu\}\otimes\{\beta\}].$$

The proof is as follows. Suppose that (ν) is a partition of n. Let S_r denote the sum of the rth powers of the characteristic roots of $A^{\{\lambda\}}$, and Z_r the sum of the rth powers of the characteristic roots of $A^{\{\mu\}}$. Then the sum of rth powers of the characteristic roots of the direct product is $S_r Z_r$.

Let ρ denote the class $(1^a\,2^b\,3^c\,\ldots)$ of the symmetric group of order $n!$, and let

$$S_\rho = S_1^a S_2^b S_3^c\ldots, \quad\text{and}\quad Z_\rho = Z_1^a Z_2^b Z_3^c\ldots.$$

Then $[\{\lambda\}\{\mu\}]\otimes\{\nu\}$ being the spur of the invariant matrix corresponding to (ν) of the direct product of $A^{\{\lambda\}}$ and $A^{\{\mu\}}$ it follows that

$$[\{\lambda\}\{\mu\}]\otimes\{\nu\} = \frac{1}{n!}\Sigma\chi_\rho^\nu S_\rho Z_\rho.$$

Thus if $\{\alpha\}'$ denotes an S-function of the characteristic roots of $A^{\{\lambda\}}$ then

$$\{\alpha\}' = \{\lambda\}\otimes\{\alpha\}$$

and $$S_\rho = \Sigma\chi_\rho^{(\alpha)}\{\alpha\}' = \Sigma\chi_\rho^{(\alpha)}\{\lambda\}\otimes\{\alpha\}.$$

Similarly $$Z_\rho = \Sigma\chi_\rho^{(\beta)}\{\mu\}\otimes\{\beta\},$$

and thus
$$\begin{aligned}[\{\lambda\}\{\mu\}]\otimes\{\nu\} &= \frac{1}{n!}\Sigma\chi_\rho^\nu S_\rho Z_\rho\\ &= \frac{1}{n!}\Sigma\chi_\rho^{(\nu)}\chi_\rho^{(\alpha)}\chi_\rho^{(\beta)}[\{\lambda\}\otimes\{\alpha\}]\,[\{\mu\}\otimes\{\beta\}]\\ &= \frac{1}{n!}\Sigma\chi_\rho^{(\nu)}K_{\alpha\beta\xi}\chi_\rho^{(\xi)}[\{\lambda\}\otimes\{\alpha\}]\,[\{\mu\}\otimes\{\beta\}]\\ &= \Sigma K_{\alpha\beta\nu}[\{\lambda\}\otimes\{\alpha\}]\,[\{\mu\}\otimes\{\beta\}].\end{aligned}$$

To illustrate these theorems $\{51\}\otimes\{2\}$ is evaluated. It is easily established, and will be proved later that

$$\{n\}\otimes\{2\} = \{2n\}+\{2n-2, 2\}+\{2n-4, 4\}+\dots$$

to $\frac{1}{2}(n+1)$ or $\frac{1}{2}(n+2)$ terms.

Also $$\{n\}\otimes\{1^2\} = \{2n-1, 1\}+\{2n-3, 3\}+\dots$$

to $\frac{1}{2}(n+1)$ or $\frac{1}{2}n$ terms.

From theorem (V) we have

$$\begin{aligned}[\{5\}\{1\}]\otimes\{2\} &= [\{5\}\otimes\{2\}]\,[\{1\}\otimes\{2\}]+[\{5\}\otimes\{1^2\}]\,[\{1\}\otimes\{1^2\}]\\ &= [\{10\}+\{82\}+\{64\}]\,\{2\}+[\{91\}+\{73\}+\{55\}]\,\{1^2\}\\ &= \{12\}+\{11.1\}+2\{10.2\}+\{93\}+\{921\}+2\{84\}\\ &\quad +\{831\}+\{822\}+\{75\}+\{741\}+\{66\}+\{651\}+\{642\}\\ &\quad +\{10.2\}+\{10.1^2\}+\{921\}+\{91^3\}+\{84\}+\{831\}\\ &\quad +\{741\}+\{731^2\}+\{6^2\}+\{651\}+\{5^2\,1^2\}.\end{aligned}$$

But $$\begin{aligned}[\{5\}\{1\}]\otimes\{2\} &= [\{6\}+\{51\}]\otimes\{2\}\\ &= \{6\}\otimes\{2\}+\{6\}\{51\}+\{51\}\otimes\{2\}\\ &= \{12\}+\{10.2\}+\{84\}+\{6^2\}+\{11.1\}+\{10.2\}\\ &\quad +\{10.1^2\}+\{93\}+\{921\}+\{84\}+\{831\}\\ &\quad +\{75\}+\{741\}+\{651\}+\{51\}\otimes\{2\}.\end{aligned}$$

Hence $$\begin{aligned}\{51\}\otimes\{2\} &= \{10.2\}+\{921\}+\{91^3\}+\{84\}\\ &\quad +\{831\}+\{82^2\}+\{741\}+\{731^2\}\\ &\quad +\{6^2\}+\{651\}+\{642\}+\{5^2\,1^2\}.\end{aligned}$$

This result gives the types of concomitants of degree 2 of a ground form of type $\{51\}$. It is not the easiest method of evaluation but does illustrate the theorems.

Application to concomitants under restricted groups of transformations

Suppose now that the group of transformations considered is restricted to those which leave a certain form or forms invariant. Such a group will be a subgroup of the full linear group, and hence any simple representation of the full linear group will be a representation, usually compound, of the subgroup. If $\{\lambda\}$ denotes an S-function, i.e. a character of the full linear group, and $[\mu]$ denotes a character of the subgroup, then an equation of the form

$$\{\lambda\} = \Sigma K_{\lambda\mu}[\mu]$$

is obtained.

Correspondingly, a tensor which is simple over the full linear group and of type $\{\lambda\}$ will in general be compound over the subgroup, and will have a simple subtensor of type $[\mu]$ for each term in the summation.

There will be a simple algebraic form, over the restricted group, for each simple character $[\mu]$. To find the concomitants of such a form of degree n in the coefficients the nth induced matrix of the matrix of transformation must be expressed as a direct sum of irreducible representations. Correspondingly, the equation

$$[\mu] \otimes \{n\} = \Sigma[\nu],$$

and the ground form of type $[\mu]$ has a concomitant of degree n and type $[\nu]$ for each term in the summation.

For the orthogonal group (Littlewood 1940, p. 240) each character $[\mu]$ may be expressed linearly in terms of S-functions by the formula

$$[\mu] = \{\mu\} + \Sigma(-1)^{\frac{1}{2}p}\, g_{\gamma\eta\mu}\{\eta\}.$$

The new multiplication may then be effected by means of formulae obtained earlier in this paper. It is hoped to elaborate this in another paper.

The construction of concomitants

The above theory determines the number and type of the concomitants of any given degree in the coefficients. When this is known the actual concomitants can be determined by the following method.

Consider first the concomitants which are linear in each of two ground forms of type $\{\lambda\}$ and $\{\mu\}$ respectively. Let the tensors of coefficients be respectively A_α and B_β, where α and β each denote a set of cogredient suffixes.

The above theory shows that the required concomitants are those which correspond to the S-functions appearing in the product $\{\lambda\}\{\mu\}$. There exists an isomorphism between the multiplication of S-functions and the multiplication of irreducible idempotents of the Frobenius algebra corresponding to different sets of symbols (Littlewood 1940, p. 91).

The tensor A_α corresponds to a substitutional expression of the group of permutations of the suffixes (α), which is an irreducible idempotent of the Frobenius algebra. The same is true of the tensor B_β and the suffixes (β). It follows that the product of the tensors $A_\alpha B_\beta$ is a complex tensor which has a simple component corresponding to each S-function which appears in the product $\{\lambda\}\{\mu\}$.

Hence the concomitants of two simple forms $A_\alpha x^\alpha$, $B_\beta x^\beta$, x^α and x^β denoting the appropriate Clebsch variable tensors, which are linear in the coefficients of each ground form, are obtained by expressing the product of the tensors $A_\alpha B_\beta$ as a sum of simple tensors and contracting each simple tensor with the appropriate Clebsch variable tensor.

If $\{\nu\}$ is one of the S-functions appearing in the product $\{\lambda\}\{\mu\}$, then that portion of the tensor $A_\alpha B_\beta$ which corresponds to the partition (ν) can be picked out by multiplying and contracting with the appropriate Clebsch tensor. The process automatically annihilates the other components of the tensor.

Let T and T' be the tableaux corresponding to A_α and B_β, and let e and e' be the corresponding irreducible idempotents. Let the $f^{(\nu)} = \chi_0^{(\nu)}$ standard tableaux corresponding to (ν) be $T_1, T_2, \ldots, T_f$, and let the irreducible idempotents be N_1P_1, N_2P_2, …, N_fP_f. It is convenient to take the order NP for the factors instead of the order PN as used by Young. Then ee' has a representation matrix in the subalgebra corresponding to (ν) of the Frobenius algebra, of which the rank r is equal to the coefficient of $\{\nu\}$ in $\{\lambda\}\{\mu\}$. Hence at least r of the tableaux T_i are such that the expression $ee'N_iP_i$ differs from zero. These are the tableaux which give the concomitants of type $\{\nu\}$.

The rules for the multiplication of S-functions depends on the construction of such tableaux for which $ee'N_iP_i$ differs from zero. Hence, *following the rules for the multiplication of the S-functions $\{\lambda\}\{\mu\}$ we actually write down the tableaux which indicate the appropriate Clebsch variable tensors to form the concomitants.*

By way of illustration the concomitants are found which are linear in each of two ground forms in four variables, one being the quadratic $a_{ij}x^ix^j$, and the other a form of type $\{21\}$, $b_{ijk}x^ix^{jk}$.

In writing down our tableaux it is noted that the two suffixes i and j are symmetrical in a_{ij}, and a repeated symbol α is used. Corresponding to the second form, the two symbols in the same row are symmetrically permuted by the corresponding factor P, and for this reason p is chosen as the right-hand factor. Hence for these the same symbol β is used. The two tableaux are thus

$$(\alpha\ \alpha), \quad \begin{pmatrix}\beta & \beta \\ \gamma & \end{pmatrix}.$$

It is easier to multiply the second by the first, so β, γ, α is taken to be the assigned order of the symbols. The product of the S-functions corresponds to the set of tableaux

$$\begin{pmatrix}\beta & \beta & \alpha & \alpha \\ \gamma & & & \end{pmatrix}, \quad \begin{pmatrix}\beta & \beta & \alpha \\ \gamma & \alpha & \end{pmatrix}, \quad \begin{pmatrix}\beta & \beta & \alpha \\ \gamma & & \\ \alpha & & \end{pmatrix}, \quad \begin{pmatrix}\beta & \beta \\ \gamma & \alpha \\ \alpha & \end{pmatrix}.$$

Hence there are four concomitants which correspond to these tableaux.

In terms of tensors these concomitants can be expressed as

$$a_{\alpha_1\alpha_2}b_{\beta_1\beta_2\gamma}x^{\beta_1\gamma}x^{\beta_2}x^{\alpha_1}x^{\alpha_2}, \quad a_{\alpha_1\alpha_2}b_{\beta_1\beta_2\gamma}x^{\beta_1\gamma}x^{\beta_2\alpha_1}x^{\alpha_2},$$

$$a_{\alpha_1\alpha_2}b_{\beta_1\beta_2\gamma}x^{\beta_1\gamma\alpha_1}x^{\beta_2}x^{\alpha_2}, \quad a_{\alpha_1\alpha_2}b_{\beta_1\beta_2\gamma}x^{\beta_1\gamma\alpha_1}x^{\beta_2\alpha_2}.$$

The symbolic expression for the concomitants is even more obvious from the tableaux. If the forms are represented by $(\alpha_ix^i)^2$ which is denoted by α_x^2, and $\beta_x(\beta\gamma\,|\,xy)$, where $\beta_x = \beta_ix^i$ and $(\beta\gamma\,|\,xy) = (\beta_i\gamma_j - \beta_j\gamma_i)\,(x^iy^j - x^jy^i)$, then the concomitants are

$$(\beta\gamma\,|\,xy)\,\beta_x\alpha_x^2, \qquad (\beta\gamma\,|\,xy)\,(\beta\alpha\,|\,xy)\,\alpha_x,$$

$$(\beta\gamma\alpha\,|\,xyz)\,\beta_x\alpha_x, \quad (\beta\gamma\alpha\,|\,xyz)\,(\beta\alpha\,|\,xy).$$

It should be noted here that the use of other standard tableaux all lead to a zero result, while non-standard tableaux lead to results which are linearly dependent on the results obtained from standard tableaux. In the general case, however, the *lattice permutation* rule (Littlewood 1940, p. 94) in the multiplication of S-functions picks out one only from a set of standard tableaux which lead to equal forms.

Concomitants which are linear in three or more ground forms are treated similarly. The product of the three or more S-functions is obtained, and the tableau obtained for each S-function in the product defines the corresponding concomitant.

Returning to the case of two ground forms, before considering the case when these are identical, so as to give the concomitants of degree 2 in one ground form, examine now the case when they correspond to the same partition (λ).

In this case the concomitants correspond to the product $\{\lambda\}\{\lambda\}$. Now

$$\{\lambda\}\{\lambda\} = \{\lambda\}\otimes\{2\}+\{\lambda\}\otimes\{1^2\},$$

and the set of concomitants are divided into two portions corresponding to $\{\lambda\}\otimes\{2\}$ and $\{\lambda\}\otimes\{1^2\}$, which are respectively symmetric and antisymmetric in the two ground forms.

Thus considering two quadratics $a_{ij}x^ix^j$, $b_{ij}x^ix^j$, then

$$\{2\}\{2\} = \{4\}+\{31\}+\{2^2\} \quad \text{and} \quad \{2\}\otimes\{2\} = \{4\}+\{2^2\},\ \{2\}\otimes\{1^2\} = \{31\}.$$

Of the three concomitants linear in the two quadratics

$$a_{ij}b_{km}x^ix^jx^kx^m \quad \text{and} \quad a_{ij}b_{km}x^{ik}x^{jm}$$

are symmetric in the two ground forms, while

$$a_{ij}b_{km}x^{ik}x^jx^m$$

is antisymmetric.

If now the two ground forms are made identical, those concomitants which are antisymmetric become equal to zero.

Hence, *the concomitants of degree* 2 *in a single ground form of type* $\{\lambda\}$ *are obtained by selecting from the tableaux corresponding to the product* $\{\lambda\}\{\lambda\}$ *those tableaux which correspond to concomitants of two ground forms each of type* $\{\lambda\}$, *subsequently made identical, which are not changed in sign when the ground forms are interchanged.*

Concomitants of degree 3 are obtained as follows. Having obtained the tableaux which correspond to $\{\lambda\}\otimes\{2\}$ we multiply again by $\{\lambda\}$. The resulting S-functions correspond to

$$\begin{aligned}[\{\lambda\}\otimes\{2\}]\{\lambda\} &= \{\lambda\}\otimes[\{2\}\{1\}] \\ &= \{\lambda\}\otimes[\{3\}+\{21\}] \\ &= \{\lambda\}\otimes\{3\}+\{\lambda\}\otimes\{21\}.\end{aligned}$$

The tableaux corresponding to $\{\lambda\}\otimes\{3\}$ give the concomitants of degree 3. The tableaux corresponding to $\{\lambda\}\otimes\{21\}$ must be rejected as leading to a zero result.

The generalization to concomitants of degree n is obvious. Corresponding to the equation

$$\{n-1\}\{1\} = \{n\}+\{n-1, 1\}$$

it is seen that

$$[\{\lambda\}\otimes\{n-1\}]\{\lambda\} = \{\lambda\}\otimes\{n\}+\{\lambda\}\otimes\{n-1, 1\}.$$

The chief difficulty is to determine which tableaux should be rejected as belonging to $\{\lambda\}\otimes\{n-1, 1\}$. If the expression for $\{\lambda\}\otimes\{n\}$ can be computed by other means, it is only necessary to pick out the tableaux of the correct types, but here again, if the same S-function appears both in $\{\lambda\}\otimes\{n\}$ and $\{\lambda\}\otimes\{n-1, 1\}$, closer examination is necessary. It is always possible, however, to find the tableau which yields a non-zero result by proceeding with the evaluation until it is assured that a non-zero result will be obtained.

On the other hand, if simple methods could be devised by substitutional analysis or otherwise, of discriminating the tableaux pertaining to $\{\lambda\}\otimes\{n\}$, then a method of computation for $\{\lambda\}\otimes\{n\}$ is found. It was hoped that the following method would meet the situation, but many difficulties arise and it has not so far been possible to overcome them.

It is quite easy to show that

$$\{n\}\otimes\{2\} = \{2n\}+\{2n-2, 2\}+\{2n-4, 4\}+\dots \text{ to } \tfrac{1}{2}(n+1) \text{ or } \tfrac{1}{2}(n+2) \text{ terms.}$$

For
$$\{n\}\{n\} = \{2n\}+\{2n-1, 1\}+\{2n-2, 2\}+\dots+\{n, n\},$$

and if the tableau corresponding to $\{2n-r, r\}$ is considered, it is clear that the interchange of the set of symbols from the first tableau with the set of symbols from the second will necessitate r interchanges in columns and $(n-r)$ interchanges in the first row. Each column interchange will contribute a minus sign, and it follows that the tableau will belong to $\{n\}\otimes\{2\}$ if and only if r is even.

Proceeding to more general cases consider first $\{21\}\otimes\{2\}$. The product $\{21\}\{21\}$ is first evaluated and the following tableaux obtained:

$$\begin{pmatrix}\alpha & \alpha & \alpha' & \alpha' \\ \beta & \beta' & & \end{pmatrix}, \quad \begin{pmatrix}\alpha & \alpha & \alpha' & \alpha' \\ \beta & & & \\ \beta' & & & \end{pmatrix}, \quad \begin{pmatrix}\alpha & \alpha & \alpha' \\ \beta & \alpha' & \beta' \end{pmatrix}, \quad \begin{pmatrix}\alpha & \alpha & \alpha' \\ \beta & \alpha' & \\ \beta' & & \end{pmatrix},$$

$$\begin{pmatrix}\alpha & \alpha & \alpha' \\ \beta & \beta' & \\ \alpha' & & \end{pmatrix}, \quad \begin{pmatrix}\alpha & \alpha & \alpha' \\ \beta & & \\ \alpha' & & \\ \beta' & & \end{pmatrix}, \quad \begin{pmatrix}\alpha & \alpha \\ \beta & \alpha' \\ \alpha' & \beta' \end{pmatrix}, \quad \begin{pmatrix}\alpha & \alpha \\ \beta & \alpha' \\ \alpha' & \\ \beta' & \end{pmatrix}.$$

Corresponding to each tableau the irreducible idempotent contains one group element which interchanges the first set of symbols α, α, β with the second set α', α', β'. It is reasonable to suppose that the coefficient will be positive or negative according as the tableau belongs to $\{21\}\otimes\{2\}$ or $\{21\}\otimes\{1^2\}$. Three interchanges are necessary, and if these can be arranged in rows or columns, the column interchanges will each contribute a minus sign.

On this reasoning the 1st, 4th and 6th tableaux should belong to $\{21\}\otimes\{2\}$, and the 2nd, 3rd and 5th to $\{21\}\otimes\{1^2\}$. The last tableau requires closer attention.

In order to replace

$$\begin{pmatrix}\alpha, & \alpha \\ \beta, & \alpha' \\ \alpha', & \beta'\end{pmatrix} \quad \text{by} \quad \begin{pmatrix}\alpha', & \alpha' \\ \beta', & \alpha \\ \alpha, & \beta\end{pmatrix}$$

two cyclic permutations of the columns are made, namely, to

$$\begin{pmatrix}\alpha' & \alpha' \\ \alpha & \beta' \\ \beta & \alpha\end{pmatrix},$$

followed by two interchanges in the second and third rows. The corresponding sign would be positive and it is deduced that

$$\{21\}\otimes\{2\} = \{42\}+\{321\}+\{31^3\}+\{2^3\}.$$

A check is desirable. The tensor corresponding to $\{21\}$ in four variables has 20 components. The number of rows in the second induced matrix will thus be $\frac{1}{2}\,.\,20\,.\,21 = 210$. The orders of the tensors on the right are given by $\{42\} = 126$, $\{321\} = 64$, $\{31^3\} = 10$, $\{2^3\} = 10$, and the equation

$$210 = 126 + 64 + 10 + 10$$

provides the check.

However, difficulties very soon arise. Thus for $\{32\} \otimes \{2\}$ there is a tableau

$$\begin{pmatrix} \alpha & \alpha & \alpha & \alpha' \\ \beta & \beta & \alpha' & \beta' \\ \alpha' & \beta' & & \end{pmatrix}.$$

To interchange the α's and β's with the α''s and β''s would appear to require three column interchanges and two row interchanges, and the tableau would appear to belong to $\{32\} \otimes \{1^2\}$. On the other hand, there is another tableau obtained by suspending the lattice permutation rule which yields the same concomitant, namely,

$$\begin{pmatrix} \alpha & \alpha & \alpha & \alpha' \\ \beta & \beta & \beta' & \beta' \\ \alpha' & \alpha' & & \end{pmatrix}.$$

This, however, would require two column and three row interchanges.

Consideration of the orders of tensors, and other methods for the computation of $\{\lambda\} \otimes \{n\}$, which will be described in Part III of this paper, show quite definitely that the tableau really belongs to $\{32\} \otimes \{2\}$.

PART III. ILLUSTRATIVE EXAMPLES

Concomitant types

Problems concerning the concomitants which are linear in the coefficients of each of any given set of ground forms may be taken as effectively solved by the methods of this paper. This includes the cases of *concomitant types* and also *perpetuants*.

As an example we obtain the concomitant types for three cubics in four variables. The cubics are taken symbolically as $(\alpha_i x^i)^3$, $(\beta_i x^i)^3$ and $(\gamma_i x^i)^3$.

Corresponding to $\{3\}\{3\}$ the tableaux are:

$$(\alpha\,\alpha\,\alpha\,\beta\,\beta\,\beta), \quad \begin{pmatrix} \alpha\,\alpha\,\alpha\,\beta\,\beta \\ \beta \end{pmatrix}, \quad \begin{pmatrix} \alpha\,\alpha\,\alpha\,\beta \\ \beta\,\beta \end{pmatrix}, \quad \begin{pmatrix} \alpha\,\alpha\,\alpha \\ \beta\,\beta\,\beta \end{pmatrix},$$

and for the product $\{3\}\{3\}\{3\}$

$$(\alpha\,\alpha\,\alpha\,\beta\,\beta\,\beta\,\gamma\,\gamma\,\gamma), \quad \begin{pmatrix} \alpha\,\alpha\,\alpha\,\beta\,\beta\,\beta\,\gamma\,\gamma \\ \gamma \end{pmatrix}, \quad \begin{pmatrix} \alpha\,\alpha\,\alpha\,\beta\,\beta\,\beta\,\gamma \\ \gamma\,\gamma \end{pmatrix}, \quad \begin{pmatrix} \alpha\,\alpha\,\alpha\,\beta\,\beta\,\beta \\ \gamma\,\gamma\,\gamma \end{pmatrix},$$

$$\begin{pmatrix} \alpha\,\alpha\,\alpha\,\beta\,\beta\,\gamma\,\gamma\,\gamma \\ \beta \end{pmatrix}, \quad \begin{pmatrix} \alpha\,\alpha\,\alpha\,\beta\,\beta\,\gamma\,\gamma \\ \beta\,\gamma \end{pmatrix}, \quad \begin{pmatrix} \alpha\,\alpha\,\alpha\,\beta\,\beta\,\gamma\,\gamma \\ \beta \\ \gamma \end{pmatrix}, \quad \begin{pmatrix} \alpha\,\alpha\,\alpha\,\beta\,\beta\,\gamma \\ \beta\,\gamma\,\gamma \end{pmatrix},$$

$$\begin{pmatrix} \alpha\,\alpha\,\alpha\,\beta\,\beta\,\gamma \\ \beta\,\gamma \\ \gamma \end{pmatrix}, \quad \begin{pmatrix} \alpha\,\alpha\,\alpha\,\beta\,\beta \\ \beta\,\gamma\,\gamma\,\gamma \end{pmatrix}, \quad \begin{pmatrix} \alpha\,\alpha\,\alpha\,\beta\,\beta \\ \beta\,\gamma\,\gamma \\ \gamma \end{pmatrix}, \quad \begin{pmatrix} \alpha\,\alpha\,\alpha\,\beta\,\gamma\,\gamma\,\gamma \\ \beta\,\beta \end{pmatrix},$$

$$\begin{pmatrix}\alpha\,\alpha\,\alpha\,\beta\,\gamma\,\gamma\\ \beta\,\beta\,\gamma\end{pmatrix},\quad \begin{pmatrix}\alpha\,\alpha\,\alpha\,\beta\,\gamma\,\gamma\\ \beta\,\beta\\ \gamma\end{pmatrix},\quad \begin{pmatrix}\alpha\,\alpha\,\alpha\,\beta\,\gamma\\ \beta\,\beta\,\gamma\,\gamma\end{pmatrix},\quad \begin{pmatrix}\alpha\,\alpha\,\alpha\,\beta\,\gamma\\ \beta\,\beta\,\gamma\\ \gamma\end{pmatrix},$$

$$\begin{pmatrix}\alpha\,\alpha\,\alpha\,\beta\,\gamma\\ \beta\,\beta\\ \gamma\,\gamma\end{pmatrix},\quad \begin{pmatrix}\alpha\,\alpha\,\alpha\,\beta\\ \beta\,\beta\,\gamma\,\gamma\\ \gamma\end{pmatrix},\quad \begin{pmatrix}\alpha\,\alpha\,\alpha\,\beta\\ \beta\,\beta\,\gamma\\ \gamma\,\gamma\end{pmatrix},\quad \begin{pmatrix}\alpha\,\alpha\,\alpha\,\gamma\,\gamma\,\gamma\\ \beta\,\beta\,\beta\end{pmatrix},$$

$$\begin{pmatrix}\alpha\,\alpha\,\alpha\,\gamma\,\gamma\\ \beta\,\beta\,\beta\\ \gamma\end{pmatrix},\quad \begin{pmatrix}\alpha\,\alpha\,\alpha\,\gamma\\ \beta\,\beta\,\beta\\ \gamma\,\gamma\end{pmatrix},\quad \begin{pmatrix}\alpha\,\alpha\,\alpha\\ \beta\,\beta\,\beta\\ \gamma\,\gamma\,\gamma\end{pmatrix}.$$

From these tableaux can be written down the 23 concomitants; for example, corresponding to

$$\begin{pmatrix}\alpha\,\alpha\,\alpha\,\beta\\ \beta\,\beta\,\gamma\\ \gamma\,\gamma\end{pmatrix}$$

the concomitant is, symbolically,

$$(\alpha\beta\gamma \mid xyz)^2\,(\alpha\gamma \mid xy)\,\beta_x.$$

As the number of ground forms increases, the number of concomitants increases very rapidly, and the evaluation of each concomitant becomes impracticable. Generating functions which give the number of concomitants of each type can, however, be obtained.

A generating function is now sought that will give the number of concomitants of each type, which are linear in each of p ground forms all of type $\{\lambda\}$ in n variables.

In virtue of the preceding theory, if

$$\{\lambda\}^p = \Sigma\{\mu\},$$

then there will be a concomitant of type $\{\mu\}$ for every term $\{\mu\}$ in the summation.

If $\{\lambda\}$ is an S-function of the quantities $x_1, x_2, \ldots, x_r$, then

$$\{\lambda\} = \frac{|\,x_s^{\lambda_t+n-t}\,|}{|\,x_s^{n-t}\,|},$$

and thus

$$\frac{|\,x_s^{\lambda_t+n-t}\,|^p}{|\,x_s^{n-t}\,|^p} = \Sigma\frac{|\,x_s^{\mu_t+n-t}\,|}{|\,x_s^{n-t}\,|}.$$

Hence the coefficient of $x_1^{\mu_1+n-1}\,x_2^{\mu_2+n-2}\ldots x_r^{\mu_r}$ in the expansion of

$$\frac{|\,x_s^{\lambda_t+n-t}\,|^p}{|\,x_s^{n-t}\,|^{p-1}}$$

is equal to the number of concomitants of type $\{\mu_1, \mu_2, \ldots, \mu_n\}$. This is the required generating function.

Thus the concomitants linear in each of p cubics in three variables have the generating function

$$\begin{vmatrix}1 & 1 & 1\\ x^5 & x & 1\\ y^5 & y & 1\end{vmatrix}^p \div \begin{vmatrix}1 & 1 & 1\\ x^2 & x & 1\\ y^2 & y & 1\end{vmatrix}^{p-1} = \frac{(y^5+x^5y+x-x^5-y^5x-y)^p}{[(x-y)\,(x-1)\,(y-1)]^{p-1}}$$

$$= \Sigma(x^{\mu_1+2}y^{\mu_2+1}+y^{\mu_1+2}x^{\mu_3}+x^{\mu_2+1}y^{\mu_3}-y^{\mu_1+2}x^{\mu_2+1}-x^{\mu_1+2}y^{\mu_3}-y^{\mu_2+1}x^{\mu_3}).$$

Thus, assuming that $\mu_1 \geqslant \mu_2 \geqslant 3p-\mu_1-\mu_2$, the number of concomitants of type

$$\{\mu_1, \mu_2, 3p-\mu_1-\mu_2\}$$

is equal to the coefficients of $x_1^{\mu_1+2} y_1^{\mu_2+1}$.

Perpetuants

The theory of *perpetuants* deals with the number of covariant types for a quantic of large order. The theory shows that if $p \geqslant r$, the number of covariants of weight r which are linear in each of a set of i n-ary p-ics is independent of p. Young (1924) has given a generating function for ternary perpetuants.

Since perpetuants are concomitant types, the present methods are immediately applicable. A generating function is now acquired which is not only simpler than that obtained by Young, but is also more general, for it gives, besides the covariants, also the mixed concomitants.

The ground forms are of type $\{p\}$. Any concomitant which is linear in i ground forms will be of type $\{ip-\lambda_1-\lambda_2, \lambda_1, \lambda_2\}$ with $ip-\lambda_1-\lambda_2 \geqslant \lambda_1 \geqslant \lambda_2$. This is called a concomitant of type (λ_1, λ_2). The covariants of weight r are given by $\lambda_1 = \lambda_2 = r$. The fundamental restriction on the covariants which are regarded as perpetuants is that $p \geqslant r$. When the result is generalized to include mixed concomitants, the corresponding restriction is taken to be $p \geqslant \lambda_1$. It will be shown that, in similarity with the result for perpetuant *covariants*, the number of concomitants of type (λ_1, λ_2) for $p \geqslant \lambda_1$ is independent of p.

Suppose that $\{\lambda\} \equiv \{ip-\lambda_1-\lambda_2, \lambda_1, \lambda_2\}$ is an S-function of the quantities $1, \alpha x, \beta x$. Then

$$\{\lambda\} \begin{vmatrix} 1, & 1, & 1 \\ \alpha^2 x^2, & \alpha x, & 1 \\ \beta^2 x^2, & \beta x, & 1 \end{vmatrix} = \begin{vmatrix} 1, & 1, & 1 \\ (\alpha x)^{ip-\lambda_1-\lambda_2+2}, & (\alpha x)^{\lambda_1+1}, & (\alpha x)^{\lambda_2} \\ (\beta x)^{ip-\lambda_1-\lambda_2+2}, & (\beta x)^{\lambda_1+1}, & (\beta x)^{\lambda_2} \end{vmatrix}.$$

To modulus x^{p+2}, this is congruent to

$$x^{\lambda_1+\lambda_2+1}(\alpha^{\lambda_1+1}\beta^{\lambda_2} - \beta^{\lambda_1+1}\alpha^{\lambda_2}),$$

which is independent of p.

Now

$$\{p\} \begin{vmatrix} 1, & 1, & 1 \\ \alpha^2 x^2, & \alpha x, & 1 \\ \beta^2 x^2, & \beta x, & 1 \end{vmatrix} = \{p\}\, x(\alpha-\beta)\,(1-\alpha x)\,(1-\beta x) = \begin{vmatrix} 1, & 1, & 1 \\ (\alpha x)^{p+2}, & \alpha x, & 1 \\ (\beta x)^{p+2}, & \beta x, & 1 \end{vmatrix}.$$

To modulus x^{p+2} this is congruent to $x(\alpha-\beta)$. Hence, expanding series formally in ascending powers of x, it is seen that

$$\{p\} \equiv 1/[(1-\alpha x)\,(1-\beta x)] \quad (\mathrm{mod}\, x^{p+1}),$$

$$\{p\}^i \equiv 1/[(1-\alpha x)\,(1-\beta x)]^i \quad (\mathrm{mod}\, x^{p+1}),$$

$$\{p\}^i \begin{vmatrix} 1, & 1, & 1 \\ (\alpha x)^2, & \alpha x, & 1 \\ (\beta x)^2, & \beta x, & 1 \end{vmatrix} \equiv \frac{x(\alpha-\beta)}{(1-\alpha x)^{i-1}\,(1-\beta x)^{i-1}} \quad (\mathrm{mod}\, x^{p+1})$$

$$\equiv \Sigma x^{\lambda_1+\lambda_2+1}\,(\alpha^{\lambda_1+1}\beta^{\lambda_2} - \beta^{\lambda_1+1}\alpha^{\lambda_2}).$$

Hence the number of concomitants of type (λ_1, λ_2) is equal to the coefficient of

$$x^{\lambda_1+\lambda_2}\alpha^{\lambda_1+1}\beta^{\lambda_2}$$

in

$$(\alpha-\beta)/(1-\alpha x)^{i-1}\,(1-\beta x)^{i-1}.$$

Now put $x = 1$, and the following result is obtained.

The number of concomitants of type $\{ip-\lambda_1-\lambda_2, \lambda_1, \lambda_2\}$ *which are linear in each of i ground forms of type* $\{p\}$ *is independent of* p *provided that* $p \geq \lambda_1$, *and is equal to the coefficient of* $\alpha^{\lambda_1+1}\beta^{\lambda_2}$ *in series obtained from*

$$(\alpha-\beta)/(1-\alpha)^{i-1}(1-\beta)^{i-1}$$

by formal expansion in ascending powers of α *and* β.

The covariants are obtained by putting $\lambda_1 = \lambda_2 = r =$ weight. Therefore the coefficient of $\alpha^{r+1}\beta^r$ must now be picked out.

It is not easy to show that the above series gives the same coefficient in the general case as Young's generating function. However, Young gives the series of coefficients explicitly for $i \leq 8$, and these prove to be identical with such explicit forms obtained from the generating function given here.

The generalization to four or more variables is quite straightforward. In four variables we take S-functions of the four quantities $1, x\alpha, x\beta, x\gamma$, and consider the series of modulus x^{p+3}. In the same manner the following result is obtained:

The number of concomitants of type $\{ip-\lambda_1-\lambda_2-\lambda_3, \lambda_1, \lambda_2, \lambda_3\}$ *which are linear in the coefficients of each of i ground forms of type* $\{p\}$ *is independent of p if* $p \geq \lambda_1$, *and equal to the coefficient of* $\alpha^{\lambda_1+2}\beta^{\lambda_2+1}\gamma^{\lambda_3}$ *in the formal expansion of*

$$\frac{(\alpha-\beta)(\alpha-\gamma)(\beta-\gamma)}{(1-\alpha)^{i-1}(1-\beta)^{i-1}(1-\gamma)^{i-1}}$$

in ascending powers of α, β, γ.

The extension to five or more variables is obvious.

Simultaneous concomitants of several ground forms

The problem remains of finding the concomitants which are of degree >1 in the coefficients of one or more of the ground forms. The corresponding formulae in S-functions involve the symbol $\otimes$, and depend on the evaluation of $\{\lambda\} \otimes \{n\}$. The evaluation of this expression corresponds to the determination of the concomitants of a single ground form of type $\{\lambda\}$.

It follows, then, that the problem of finding the simultaneous concomitants of any set of ground forms is reduced to the separate problems of finding the concomitants of single ground forms.

This is illustrated with reference to the simultaneous concomitants of a binary quadratic and a binary cubic. The concomitants of each of these forms, taken separately, are well known.

Let the quadratic be $$f = \alpha_x^2 = \beta_x^2 = \gamma_x^2$$

and the cubic $$g = \xi_x^3 = \eta_x^3 = \zeta_x^3.$$

A concomitant which is of degree p in the coefficients of the quadratic, and of degree q in the coefficients of the cubic, will be said to be of degree (p, q).

Then
$$\{2\}\otimes\{2\} = \{4\}+\{2^2\},\ \{2\}\otimes\{3\} = \{6\}+\{42\},$$
$$\{2\}\otimes\{4\} = \{8\}+\{62\}+\{4^2\},\ \{2\}\otimes\{5\} = \{10\}+\{82\}+\{64\},$$
$$\{3\}\otimes\{2\} = \{6\}+\{42\},\ \{3\}\otimes\{3\} = \{9\}+\{72\}+\{63\},$$
$$\{3\}\otimes\{4\} = \{12\}+\{10.2\}+\{93\}+\{84\}+\{6^2\}.$$

The only irreducible concomitant of the quadratic is the invariant D of degree $(2, 0)$ and type $\{2^2\}$. The irreducible concomitants of the cubic are P, of degree $(0, 2)$ and type $\{42\}$, Q of degree $(0, 3)$ and type $\{63\}$, and the invariant R of degree $(0, 4)$ and type $\{6^2\}$.

The concomitants of degree $(1, 1)$ correspond to the formula

$$\{2\}\{3\} = \{5\}+\{41\}+\{32\}.$$

The term $\{5\}$ corresponds to the product of the ground forms. The other two concomitants are irreducible and are denoted by T and U respectively.

The concomitants of degree $(2, 1)$ correspond to the formula

$$\{2\}\otimes\{2\}\{3\} = [\{4\}+\{2^2\}]\{3\} = \{7\}+\{61\}+2\{52\}+\{43\}.$$

The convention is generally adopted that in such an expression as $\{2\}\otimes\{2\}\{2\}$ the symbol $\otimes$ takes precedence over ordinary multiplication and is evaluated first, just as $2\times 2+2$ is taken to mean $(2\times 2)+2$. The reducible concomitants of degree $(2, 1)$ are

$$f^2g, \quad Dg, \quad fT, \quad fU$$

of types $\{7\}, \quad \{52\}, \quad \{61\}, \quad \{52\}$

respectively.

Hence there is one irreducible concomitant of degree $(2, 1)$ which is of type $\{43\}$ and is denoted by V.

The concomitants of degree $(3, 1)$ correspond to

$$\{2\}\otimes\{3\}\{3\} = [\{6\}+\{42\}]\{3\} = \{9\}+\{81\}+2\{72\}+2\{63\}+\{54\}.$$

The reducible concomitants are

$$f^3g, \quad fGg, \quad f^2T, \quad f^2U, \quad DT, \quad DU, \quad fV.$$

The types are the same as those of the complete set of concomitants of degree $(3, 1)$, so that clearly there can be no irreducible concomitants of this degree.

It is not necessary to advance further with concomitants of degree $(n, 1)$, and degrees of the form $(n, 2)$ are now proceeded with.

The expression corresponding to the degree $(1, 2)$ is

$$\begin{aligned}\{2\}\{3\}\otimes\{2\} &= \{2\}[\{6\}+\{42\}]\\ &= \{8\}+\{71\}+2\{62\}+\{53\}+\{4^2\}.\end{aligned}$$

The reducible concomitants are

$$fg^2, \quad fP, \quad Tg, \quad Ug$$

of types respectively $\{8\}, \quad \{62\}, \quad \{71\}, \quad \{62\}.$

There are thus two irreducible concomitants of degree $(1, 2)$ which are denoted by θ, ϕ, and of types $\{53\}$ and $\{4^2\}$ respectively.

Investigation of degree $(2, 2)$ reveals no irreducible concomitants, but two syzygies. Thus

$$\begin{aligned}\{3\}\otimes\{2\}\{2\}\otimes\{2\} &= [\{6\}+\{42\}][\{4\}+\{2^2\}]\\ &= \{10\}+\{91\}+3\{82\}+2\{73\}+3\{64\}.\end{aligned}$$

There are twelve reducible concomitants which correspond to the types

$$\{10\}+\{91\}+4\{82\}+3\{73\}+3\{64\}.$$

Evidently the four reducible concomitants of type {82}, namely,

$$g^2D,\quad f^2P,\quad fgU,\quad T^2$$

are connected by a syzygy, and similarly for the three reducible concomitants of type {73}, namely,

$$TU,\quad gV,\quad f\theta.$$

In a similar manner for the degree (3, 2) an invariant T of type $\{6^2\}$ is obtained, for the degree (1, 3) a concomitant ψ of type {6, 5}, for the degree (2, 3) a concomitant Θ of type {7, 6}, and for the degree (3, 4) an invariant J of type $\{9^2\}$. Further examination reveals no other irreducible concomitant and the system

$$f,\ g,\ D,\ P,\ Q,\ R,\ T,\ U,\ V,\ \theta,\ \phi,\ \psi,\ \Theta,\ I,\ J$$

is complete (see Grace & Young 1903, p. 165).

A complete account of the syzygies could also be obtained by these methods, but this will not be attempted here.

When the system of ground forms is more complicated the work is of course more laborious, but when the set of concomitants of each ground form singly is known, it is entirely routine work except for one consideration.

The system of the binary quadratic and cubic considered above has three concomitants of degree (2, 2) and type {64}. There are three reducible concomitants of this degree and type, which we assume to give the three concomitants. But a theoretical possibility exists that the three reducible concomitants are connected by a syzygy, and thus supply only two of the linearly independent concomitants of this degree and type, the third would then be an independent irreducible concomitant. There is a theoretical possibility of such an occurrence for any set of ground forms whenever it is found that, for a given degree and type at least three linearly independent concomitants and at least three reducible concomitants exist.

It would appear to be a highly improbable occurrence that in this way a syzygy and an irreducible concomitant of the same degrees and type should exist together, but if the circumstance did occur, it could not be revealed by an S-functional analysis such as has been considered. It would be revealed, however, when the actual concomitants were obtained and examined.

The circumstance does not occur with the binary cubic and quadratic, nor with any other set of concomitants considered in this paper.

The irreducible concomitants T, U, V, θ, ϕ, ψ, of which the types have been found above, are easily obtained by building the appropriate tableaux, as there is no room for error in the method of building. They are obtained from the following tableaux:*

$$\begin{pmatrix}\xi\,\xi\,\xi\,\alpha \\ \alpha\end{pmatrix},\quad \begin{pmatrix}\xi\,\xi\,\xi \\ \alpha\,\alpha\end{pmatrix},\quad \begin{pmatrix}\xi\,\xi\,\xi\,\alpha \\ \alpha\,\beta\,\beta\end{pmatrix},\quad \begin{pmatrix}\xi\,\xi\,\xi\,\eta\,\alpha \\ \eta\,\eta\,\alpha\end{pmatrix},\quad \begin{pmatrix}\xi\,\xi\,\xi\,\eta \\ \eta\,\eta\,\alpha\,\alpha\end{pmatrix},$$

$$\begin{pmatrix}\xi\,\xi\,\xi\,\eta\,\zeta\,\zeta \\ \eta\,\eta\,\zeta\,\alpha\,\alpha\end{pmatrix},\quad \begin{pmatrix}\xi\,\xi\,\xi\,\eta\,\eta\,\eta \\ \alpha\,\alpha\,\beta\,\beta\,\gamma\,\gamma\end{pmatrix},\quad \begin{pmatrix}\xi\,\xi\,\xi\,\eta\,\zeta\,\zeta\,\zeta \\ \eta\,\eta\,\alpha\,\alpha\,\beta\,\beta\end{pmatrix},\quad \begin{pmatrix}\xi\,\xi\,\xi\,\eta\,\zeta\,\zeta\,\omega\,\omega\,\omega \\ \eta\,\eta\,\zeta\,\alpha\,\alpha\,\beta\,\beta\,\gamma\,\gamma\end{pmatrix}.$$

* Gordan's *transvectants* have similar properties to these tableaux. The chief distinction is perhaps the systematic use here of Young's standard tableaux.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Concomitants of a single ground form

As it has been shown that problems of simultaneous concomitants of several ground forms are solvable when the set of concomitants of each individual ground form is known, the problem of finding the concomitants of a single ground form is now proceeded with.

The corresponding problem in S-functions is the evaluation of $\{\lambda\}\otimes\{p\}$.

Some progress can be made simply by the counting of coefficients. The concomitants and syzygies for a binary cubic may be completely determined in this way. Denote the cubic by f.

The number of terms in a binary form of type $\{\lambda_1, \lambda_2\}$ is $\lambda_1-\lambda_2+1$. Thus

$$\{3\} = 4$$

and hence
$$\{3\}\otimes\{2\} = \frac{4\,.\,5}{1\,.\,2} = 10.$$

The terms of $\{3\}\otimes\{2\}$ are included in the terms of

$$\{3\}\{3\} = \{6\}+\{51\}+\{42\}+\{33\}.$$

The concomitants must include the square of the cubic $\{6\}$ for which $\{6\} = 7$, and since also

$$\{51\} = 5, \quad \{42\} = 3, \quad \{33\} = 1,$$

it is clear that the only method of obtaining 10 is to take $7+3$. Thus

$$\{3\}\otimes\{2\} = \{6\}+\{42\}.$$

Again
$$\{3\}\otimes\{3\} = \frac{4\,.\,5\,.\,6}{1\,.\,2\,.\,3} = 20.$$

The reducible concomitants are

$$\{9\} = 10, \quad \{72\} = 6.$$

From the equation
$$20 = 10+6+4,$$

it is deduced that
$$\{3\}\otimes\{3\} = \{9\}+\{72\}+\{63\}.$$

For degree 4
$$\{3\}\otimes\{4\} = \frac{4\,.\,5\,.\,6\,.\,7}{1\,.\,2\,.\,3\,.\,4} = 35$$

is obtained.

The reducible concomitants are

$$\begin{aligned}\{12\}+\{10\,.\,2\}+\{93\}+\{84\} &= 13+9+7+5\\ &= 34.\end{aligned}$$

It must follow that

$$\{3\}\otimes\{4\} = \{12\}+\{10\,.\,2\}+\{93\}+\{84\}+\{66\}.$$

For the degree 5 it is found that

$$\{3\}\otimes\{5\} = 56,$$

and the reducible concomitants give

$$\begin{aligned}&\{15\}+\{13\,.\,2\}+\{12\,.\,3\}+\{11\,.\,4\}+\{10\,.\,5\}+\{96\}\\ &= 16+12+10+8+6+4\\ &= 56,\end{aligned}$$

and the set is complete.

For degree 6 $\{3\}\otimes\{6\} = 84$,
and the reducible concomitants give

$$\{18\}+\{16.2\}+\{15.3\}+\{14.4\}+\{13.5\}+\{12.6\}+\{12.6\}+\{10.8\}+\{12.6\}$$
$$= 19+15+13+11+9+7+7+3+7$$
$$= 91.$$

The discrepancy is clearly caused by a syzygy connecting the three forms of type $\{12.6\}$.

With more complicated ground forms the method soon fails, there being a considerable choice of types which lead to the same order. The method does, however, provide a very valuable check on the accuracy of an evaluation of $\{\lambda\}\otimes\{p\}$ obtained by any other method.

This method of counting coefficients depends on the known formulae for the S-functions of the roots of $(x-1)^p = 0$. Also, formulae are known (Littlewood & Richardson 1933) for the S-functions of the quantities $1, \rho, \rho^2, \ldots, \rho^r$. Use can be made of these to obtain another method for the evaluation of $\{\lambda\}\otimes\{n\}$ which is illustrated with reference to the ternary cubic.

Let $A = \text{diag}\,(1, \rho, \rho^2)$, and let $\{\lambda\}$ denote an S-function of the characteristic roots of this matrix. Then

$$\{3\} = 1+\rho+2\rho^2+2\rho^3+2\rho^4+\rho^5+\rho^6,$$

and the characteristic roots of $A^{\{3\}}$ are the ten constituents of this sum.

Hence $\{3\}\otimes\{3\}$ is the sum of the powers and products of degree 3 of these ten quantities, and thus

$$\{3\}\otimes\{3\} = 1+\rho+3\rho^2+5\rho^3+9\rho^4+12\rho^5+19\rho^6+21\rho^7+\ldots.$$

It should be noticed that the expansion of

$$\{\lambda_1, \lambda_2, \lambda_3\} = \frac{\rho^{\lambda_2+2\lambda_3}(1-\rho^{\lambda_1-\lambda_3+2})\,(1-\rho^{\lambda_1-\lambda_2+1})\,(1-\rho^{\lambda_2-\lambda_3+1})}{(1-\rho)^2\,(1-\rho^2)}$$

in ascending powers of ρ commences with the term $\rho^{\lambda_2+2\lambda_3}$. Hence the first term in the expression for $\{3\}\otimes\{3\}$ must be

$$\{9\} = 1+\rho+2\rho^2+2\rho^3+3\rho^4+3\rho^5+4\rho^6+4\rho^7+\ldots.$$

The first discrepancy of this series with the series for $\{3\}\otimes\{3\}$ is in the coefficient of ρ^2, and the next to be taken is

$$\{72\} = \rho^2+2\rho^3+4\rho^4+5\rho^5+7\rho^6+8\rho^7+\ldots.$$

One more term in ρ^3 is needed, and thus

$$\{63\} = \rho^3+2\rho^4+4\rho^5+6\rho^6+7\rho^5+\ldots.$$

The sum coincides with $\{3\}\otimes\{3\}$ up to the terms in ρ^5, but an extra $2\rho^6$ is required.

The S-functions $\{\lambda_1, \lambda_2, \lambda_3\}$ corresponding to partitions of 9 for which $\lambda_2+2\lambda_3 = 6$ are $\{522\}$ and $\{441\}$. Examination shows that both of these are required and the sum then coincides with $\{3\}\otimes\{3\}$. Thus

$$\{3\}\otimes\{3\} = \{9\}+\{72\}+\{63\}+\{522\}+\{441\}.$$

The method is more definite than the counting of terms, but is still not quite specific. Also it becomes very laborious for higher degrees.

Before turning to three methods which have proved really effective, mention should be made, at this point, of the method of first principles.

Now proceed to find $\{2^2\}\otimes\{2\}$ in four variables. Let A be a diagonal matrix with diagonal elements $\alpha, \beta, \gamma, \delta$. To find the characteristic roots of $A^{\{2^2\}}$, its spur $\{2^2\}$ must be expressed as a sum of monomial symmetric functions.

If $\{\lambda\} = \Sigma K_{\lambda\mu}\alpha^{\mu_1}\beta^{\mu_2}\gamma^{\mu_3}\delta^{\mu_4}$, then $K_{\lambda\mu}$ is the number of standard Young tableaux that can be formed corresponding to the partition (λ), with the symbols $\alpha, \beta, \gamma, \delta$, repeated respectively $\mu_1, \mu_2, \mu_3, \mu_4$ times, no symbol being repeated in the same column (Littlewood 1940, p. 94).

Thus from the tableaux

$$\begin{pmatrix}\alpha & \alpha\\ \beta & \beta\end{pmatrix} \quad \begin{pmatrix}\alpha & \alpha\\ \beta & \gamma\end{pmatrix} \quad \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \quad \begin{pmatrix}\alpha & \gamma\\ \beta & \delta\end{pmatrix}$$

it is deduced that

$$\begin{aligned}\{2^2\} &= \Sigma\alpha^2\beta^2 + \Sigma\alpha^2\beta\gamma + 2\alpha\beta\gamma\delta\\ &= (22) \;\; + (21^2) \;\; + 2(1^4),\end{aligned}$$

where (p, q, r, s) denotes the corresponding monomial symmetric function. Each monomial expression is a characteristic root of $A^{\{2^2\}}$.

The characteristic roots of $[A^{\{2^2\}}]^{\{2\}}$ are the squares and products of degree 2 of the various terms in the expansion of $\{2^2\}$. Hence $\{2^2\}\otimes\{2\}$ is the sum of these squares and products which is equal to

$$\{2^2\}\otimes\{2\} = (44)+(431)+2(422)+2(4211)+2(332)+4(3311)+6(3221)+12(2222).$$

From the following tableaux is obtained the expression of $\{4^2\}$ as a monomial symmetric function:

$$\begin{pmatrix}\alpha\,\alpha\,\alpha\,\alpha\\ \beta\,\beta\,\beta\,\beta\end{pmatrix}, \quad \begin{pmatrix}\alpha\,\alpha\,\alpha\,\alpha\\ \beta\,\beta\,\beta\,\gamma\end{pmatrix}, \quad \begin{pmatrix}\alpha\,\alpha\,\alpha\,\alpha\\ \beta\,\beta\,\gamma\,\gamma\end{pmatrix}, \quad \begin{pmatrix}\alpha\,\alpha\,\alpha\,\alpha\\ \beta\,\beta\,\gamma\,\delta\end{pmatrix}, \quad \begin{pmatrix}\alpha\,\alpha\,\alpha\,\beta\\ \beta\,\beta\,\gamma\,\gamma\end{pmatrix}, \quad \begin{pmatrix}\alpha\,\alpha\,\alpha\,\beta\\ \beta\,\beta\,\gamma\,\delta\end{pmatrix},$$

$$\begin{pmatrix}\alpha\,\alpha\,\alpha\,\gamma\\ \beta\,\beta\,\beta\,\delta\end{pmatrix}, \quad \begin{pmatrix}\alpha\,\alpha\,\alpha\,\beta\\ \beta\,\gamma\,\gamma\,\delta\end{pmatrix}, \quad \begin{pmatrix}\alpha\,\alpha\,\alpha\,\gamma\\ \beta\,\beta\,\gamma\,\delta\end{pmatrix}, \quad \begin{pmatrix}\alpha\,\alpha\,\beta\,\beta\\ \gamma\,\gamma\,\delta\,\delta\end{pmatrix}, \quad \begin{pmatrix}\alpha\,\alpha\,\beta\,\gamma\\ \beta\,\gamma\,\delta\,\delta\end{pmatrix}, \quad \begin{pmatrix}\alpha\,\alpha\,\gamma\,\gamma\\ \beta\,\beta\,\delta\,\delta\end{pmatrix}.$$

Thus $\quad \{4^2\} = (44)+(431)+(422)+(4211)+(332)+2(3311)+2(3221)+3(2222).$

Similarly

$$\begin{aligned}\{42^2\} &= (422)+(4211)+(332)+(3311)+2(3221)+6(2222),\\ \{3311\} &= (3311)+(3221)+2(2222),\\ \{2222\} &= (2222).\end{aligned}$$

Thus

$$\{2^2\}\otimes\{2\} = \{4^2\}+\{42^2\}+\{3^21^2\}+\{2^4\}.$$

The method soon becomes laborious. Three much more rapid methods suitable for the general case $\{\lambda\}\otimes\{n\}$ will be described later.

The quadratic and the linear complex

The concomitants of a quadratic in any number of variables are easily obtained by picking out the tableaux which are not changed in sign for the interchange of two ground forms. The following tableaux represent the concomitants up to degree 4:

$$(\alpha\,\alpha\,\beta\,\beta), \qquad \begin{pmatrix}\alpha\,\alpha\\ \beta\,\beta\end{pmatrix}, \qquad (\alpha\,\alpha\,\beta\,\beta\,\gamma\,\gamma), \quad \begin{pmatrix}\alpha\,\alpha\,\beta\,\beta\\ \gamma\,\gamma\end{pmatrix}, \quad \begin{pmatrix}\alpha\,\alpha\\ \beta\,\beta\\ \gamma\,\gamma\end{pmatrix},$$

$$(\alpha\,\alpha\,\beta\,\beta\,\gamma\,\gamma\,\delta\,\delta), \quad \begin{pmatrix}\alpha\,\alpha\,\beta\,\beta\,\gamma\,\gamma\\ \delta\,\delta\end{pmatrix}, \quad \begin{pmatrix}\alpha\,\alpha\,\beta\,\beta\\ \gamma\,\gamma\\ \delta\,\delta\end{pmatrix}, \quad \begin{pmatrix}\alpha\,\alpha\\ \beta\,\beta\\ \gamma\,\gamma\\ \delta\,\delta\end{pmatrix}.$$

The rule is obvious from the examples. Each pair of symbols must be placed together in the same row. If a pair of symbols was separated, and these placed in different rows, e.g.

$$\begin{pmatrix} \alpha\ \alpha\ \beta\ \beta \\ \gamma\ \gamma\ \delta \\ \delta \end{pmatrix},$$

then the pair could be interchanged with another pair, in the example the γ's with the δ's, in such a way that one interchange was in a column and one in a row. The interchange would thus multiply the corresponding expression by -1, and when the ground forms were made identical the expression would become zero.

It follows that there is a concomitant of a quadratic corresponding to every partition such that every part is an even number, and in r variables the irreducible set of concomitants consist of r forms of type

$$\{2\}, \quad \{2^2\}, \quad \{2^3\}, \quad \ldots, \quad \{2^r\},$$

respectively.

The case of the linear complex is exactly similar, save for the interchange of rows and columns. Each pair of symbols must be placed in the same column, e.g.

$$\begin{pmatrix} \alpha & \beta \\ \alpha' & \beta' \end{pmatrix}, \quad \begin{pmatrix} \alpha \\ \alpha' \\ \beta \\ \beta' \end{pmatrix}, \quad \begin{pmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{pmatrix}, \quad \begin{pmatrix} \alpha & \beta \\ \alpha' & \beta' \\ \gamma \\ \gamma' \end{pmatrix}, \quad \begin{pmatrix} \alpha \\ \alpha' \\ \beta \\ \beta' \\ \gamma \\ \gamma' \end{pmatrix}.$$

The set of irreducible concomitants for a linear complex in $2r$ or $2r+1$ variables consists of r forms of the respective types

$$\{1^2\}, \quad \{1^4\}, \quad \{1^6\}, \quad \ldots, \quad \{1^{2r}\}.$$

These results have previously been obtained by other methods.

The resemblance between the sets of concomitants for the quadratic and for the linear complex constitutes a special case of the *theorem of conjugates* which will be proved later.

The cubic

The three short methods which have been mentioned for the evaluation of $\{\lambda\} \otimes \{n\}$ in the general case will be illustrated with reference to the cubic.

The first method is a method of obtaining the concomitants in $n+1$ variables when the concomitants in n variables are known. The concomitants of the binary cubic have been obtained and those of the ternary cubic of degree 2, 3, 4 in the coefficients will now be obtained.

Let A denote a matrix with two rows and columns, and let $\{\lambda\}$ denote an S-function of its characteristic roots. Let A' denote the direct sum of the matrix A and a one-rowed matrix with element x, and let $\{\lambda\}'$ denote an S-function of its characteristic roots.

Then, generally, $$\{\lambda\}' = \{\lambda\} + \Sigma g_{n\mu\lambda}\{\mu\}\, x^n,$$
where $g_{n\mu\lambda}$ is the coefficient of $\{\lambda\}$ in $\{n\}\{\mu\}$. Thus

$$\{3\}' = \{3\} + x\{2\} + x^2\{1\} + x^3.$$

Hence $\{3\}' \otimes \{3\} = \{3\} \otimes \{3\} + x[\{3\} \otimes \{2\}\{2\}] + x^2[\{3\}\{2\} \otimes \{2\} + \{3\} \otimes \{2\}\{1\}] + \ldots.$

The coefficient of x is

$$[\{6\}+\{42\}]\{2\} = \{8\}+\{71\}+\{62\}+\{62\}+\{53\}+\{44\}.$$

Since $$\{3\} \otimes \{3\} = \{9\}+\{72\}+\{63\},$$

three of the terms in $\{3\}' \otimes \{3\}$ must be

$$\{9\}'+\{72\}'+\{63\}'.$$

Then

$$\{9\}' = \{9\}+x\{8\}+x^2\{7\}+x^3\{6\}+\ldots,$$
$$\{72\}' = \{72\}+x[\{71\}+\{62\}]+x^2[\{7\}+\{61\}+\{52\}]+\ldots,$$
$$\{63\}' = \{63\}+x[\{62\}+\{53\}]+x^2[\{61\}+\{52\}+\{43\}]+\ldots.$$

The coefficient of x in the sum is

$$\{8\}+\{71\}+2\{62\}+\{53\}.$$

To use up the extra term $\{44\}$ in $\{3\}' \otimes \{3\}$ take

$$\{441\}' = x\{44\}+x^2\{43\}+x^3\{42\}+\ldots.$$

Next, the coefficient of x^2 in $\{3\}' \otimes \{3\}$ is

$$\{3\}[\{4\}+\{2^2\}]+[\{6\}+\{42\}]\{1\} = 2\{7\}+2\{61\}+3\{52\}+2\{43\},$$

and in $$\{9\}'+\{72\}'+\{63\}'+\{441\}',$$

is $$2\{7\}+2\{61\}+2\{52\}+2\{43\}.$$

The extra term $\{52\}$ must be obtained from the S-function $\{522\}'$. The counting of coefficients shows that our list is now complete and

$$\{3\}' \otimes \{3\} = \{9'\}+\{72\}'+\{63\}'+\{441\}'+\{522\}'.$$

Thus for the degree 4

$$\{3\}' \otimes \{4\} = \{3\} \otimes \{4\}+x\{3\}' \otimes \{3\}\{2\}+x^2[\{3\}' \otimes \{3\}\{1\}+\{3\}' \otimes \{2\}\{2\}' \otimes \{2\}]+\ldots.$$

The known terms are

$$\{12\}',\quad \{10.2\}',\quad \{93\}',\quad \{84\}',\quad \{66\}',\quad \{822\}',\quad \{741\}',$$

which include the concomitants from the binary cubic and also the reducible concomitants.

The coefficient of x comes to

$$\{11\}+\{10.1\}+2\{92\}+2\{83\}+2\{74\}+\{65\},$$

which is the same as that obtained from the known concomitants.

The coefficient of x^2 is

$$2\{10\}+2\{91\}+4\{82\}+4\{73\}+4\{64\},$$

which exceeds that obtained from the known concomitants by $\{73\}+\{64\}$. Therefore the two extra terms

$$\{732\}'+\{642\}'$$

must be taken.

The counting of terms gives 715 for $\{3\}' \otimes \{4\}$ and 714 for the known S-functions. There must remain a single invariant of type $\{4^3\}'$, and so

$$\{3\}' \otimes \{4\} = \{12\}' + \{10\,.\,2\}' + \{93\}' + \{84\}' + \{66\}' + \{822\}' + \{741\}' + \{732\}' + \{642\}' + \{444\}'.$$

Thus progress could be made to four variables by letting A denote a three-rowed matrix.

As before $$\{3\}' = \{3\} + x\{2\} + x^2\{1\} + x^3.$$

The coefficient of x in $\{3\}' \otimes \{4\}$ is

$$\{3\}' \otimes \{3\}\{2\} = \{11\} + \{10\,.\,1\} + 2\{92\} + 2\{83\} + \{821\} + 2\{74\} + 2\{731\} + 2\{722\} + \{65\} + 2\{641\} + 2\{632\} + 2\{542\} + \{443\}.$$

The coefficient of x in the known terms uses up all the expression except $\{542\}$. It is deduced that $\{3\}' \otimes \{4\}$ contains a term $\{5421\}'$.

Similarly the coefficient of x^2 indicates a term $\{6222\}'$, and the equation

$$\{3\}' \otimes \{4\} = \{12\}' + \{10\,.\,2\}' + \{93\}' + \{84\}' + \{66\}' + \{822\}' + \{741\}' + \{732\}' + \{642\}' + \{444\}' + \{5421\}' + \{62^3\}'$$

proves to be complete on the counting of terms.

The second method is illustrated in obtaining $\{3\} \otimes \{5\}$. The method is similar, but the series is treated from the other end. This gives the solution in one step in any number of variables. It does, however, require the equation of more coefficients than the first method.

Let A denote any matrix, xA the result of scalar multiplication by x, and A' the direct sum of the unit matrix of order 1, and xA. Denote by $\{\lambda\}$ and $\{\lambda\}'$ S-functions of the characteristic roots of A and A' respectively. Then if (λ) is a partition of p

$$\{\lambda\}' = x^p\{\lambda\} + \Sigma g_{n\mu\lambda}\, x^{p-n}\{\mu\}.$$

In particular $$\{3\}' = 1 + x\{1\} + x^2\{2\} + x^3\{3\}.$$

Hence $$\{3\}' \otimes \{5\} = 1 + x\{1\} + x^2[\{2\} + \{1\} \otimes \{2\}] + x^3[\{3\} + \{1\}\{2\} + \{1\} \otimes \{3\}] + x^4[\{3\}\{1\} + \{2\} \otimes \{2\} + \{2\}\{1\} \otimes \{2\} + \{1\} \otimes \{4\}] + \ldots.$$

There is a term in the coefficient of x^p corresponding to each partition of p into not more than five parts with no part exceeding 3. The term corresponding to $(1^a 2^b 3^c)$ is

$$\{1\} \otimes \{a\}\{2\} \otimes \{b\}\{3\} \otimes \{c\}\, x^{a+2b+3c}.$$

The first term in $\{3\}' \otimes \{5\}$ is

$$\{15\}' = 1 + x\{1\} + x^2\{2\} + x^3\{3\} + \ldots,$$

which coincides with the series for two terms. The coefficient of x^2 in $\{3\}' \otimes \{5\}$ is, however, $2\{2\}$, and to make use of the extra term $\{2\}$ it is seen that

$$\{13\,.\,2\}' = x^2\{2\} + x^3[\{3\} + \{21\}] + x^4[\{4\} + \{31\} + \{22\}] + \ldots.$$

The coefficient of x^3 in $\{3\}' \otimes \{5\}$ is

$$3\{3\} + \{21\},$$

and in $\{15\}' + \{13\,.\,2\}'$ is

$$2\{3\} + \{21\}.$$

The next term must therefore be

$$\{12\,.\,3\}' = x^3\{3\} + x^4[\{4\} + \{31\}] + \ldots.$$

The coefficient of x^4 in $\{3\}' \otimes \{5\}$ is

$$4\{4\}+2\{31\}+2\{2^2\},$$

of which the terms already obtained supply

$$3\{4\}+2\{31\}+\{2^2\}.$$

To provide the extra S-functions it is seen that

$$\{11.4\}' = x^4\{4\}+x^5[\{5\}+\{41\}]+\ldots,$$
$$\{11.2^2\} = x^4\{2^2\}+x^5[\{32\}+\{2^21\}]+\ldots.$$

The coefficient of x^5 is

$$\{3\}\{2\}+\{3\}\{1\}\otimes\{2\}+\{2\}\otimes\{2\}\{1\}+\{2\}\{1\}\otimes\{3\}+\{1\}\otimes\{5\}$$
$$= 3\{3\}\{2\}+\{5\}+\{1\}[\{4\}+\{2^2\}]$$
$$= 5\{5\}+4\{41\}+4\{32\}+\{2^21\}.$$

The known concomitants provide

$$4\{5\}+3\{41\}+3\{32\}+\{2^21\}.$$

The extra terms are therefore taken as

$$\{10.5\}'+\{10.41\}'+\{10.32\}'.$$

This should be sufficient to explain the method. Instead of completing $\{3\}\otimes\{5\}$ by the use of it, we proceed to the third method. This last method is very rapid and effective, though it shows certain drawbacks when the degrees become large. Following the method for consecutive values of the first part, at times a choice of certain alternatives is presented. Choosing the correct partitions the work develops rapidly to the correct conclusion. But if a wrong partition is chosen the working goes further and further astray until the error is apparent. The ability to complete the working is a guarantee that no error or omission has been made. But with the larger degrees much time can be lost in following false tracks. It is an advantage to combine this with other methods to give some guidance in the choices presented.

Reverting to the nomenclature of the first method it is seen that

$$A' = A \dot{+} x,$$

so that $$\{\lambda\}' = \{\lambda\}+x\,\Sigma g_{1\mu\lambda}\{\mu\}+\text{higher powers of } x,$$

where $g_{1\mu\lambda}$ is the coefficient of $\{\lambda\}$ in $\{1\}\{\mu\}$. Then

$$\{\lambda\}'\otimes\{n\} = \{\lambda\}\otimes\{n\}+x\{\lambda\}\otimes\{n-1\}\,\Sigma g_{1\mu\lambda}\{\mu\}+\text{higher powers of } x.$$

Hence if $$\{\lambda\}\otimes\{n\} = \Sigma\{\nu\},$$

so that $$\{\lambda\}'\otimes\{n\} = \Sigma\{\nu\}'$$
$$= \sum_{\nu}[\{\nu\}+x\,\Sigma g_{1\mu\nu}\{\mu\}+\text{higher powers of } x].$$

Equating coefficients of x the following result is obtained:

Theorem. *If* $$\{\lambda\}\otimes\{n\} = \Sigma\{\nu\},$$

then $$\Sigma g_{1\zeta\nu}\{\zeta\} = \{\lambda\}\otimes\{n-1\}[\Sigma g_{1\mu\lambda}\{\mu\}].$$

Taking $\{\lambda\} = \{3\}$, then $\Sigma g_{1\mu\lambda}\{\mu\} = \{2\}$, and thus if

$$\{3\} \otimes \{5\} = \Sigma\{\nu\},$$

then

$$\Sigma g_{1\zeta\nu}\{\zeta\} = [\{3\} \otimes \{4\}]\,\{2\}$$

$$= [\{12\}+\{10.2\}+\{93\}+\{84\}+\{66\}+\{82^2\}+\{741\}+\{732\}+\{642\}+\{4^3\}+\{5421\}+\{62^3\}]\,\{2\}$$

$$= \{14\}+\{13.1\}+2\{12.2\}+2\{11.3\}+\{11.21\}+3\{10.4\}+2\{10.31\}+2\{10.2^2\}+2\{95\}$$
$$+3\{941\}+3\{932\}+\{92^21\}+2\{86\}+2\{851\}+5\{842\}+\{841^2\}+\{833\}+2\{8321\}+2\{82^3\}$$
$$+2\{761\}+3\{752\}+\{751^2\}+3\{743\}+4\{7421\}+\{73^21\}+2\{732^2\}+\{72^31\}+2\{6^22\}+\{653\}$$
$$+2\{6521\}+2\{64^2\}+2\{6431\}+3\{642^2\}+\{6421^2\}+\{632^21\}+\{62^4\}+\{5^231\}+\{5^22^2\}$$
$$+\{5^221^2\}+2\{54^21\}+\{5432\}+\{5431^2\}+\{542^21\}+\{4^32\}.$$

It is required to pick out the S-functions $\{\nu\}$ such that $\Sigma g_{1\zeta\nu}\{\zeta\}$ is included in the above list.

Clearly

$$\{15\} \rightarrow \{14\},$$
$$\{13.2\} \rightarrow \{13.1\}+\{12.2\},$$
$$\{12.3\} \rightarrow \{12.2\}+\{11.3\}.$$

This exhausts the partitions commencing with 12. Beginning with 11, $\{11.3\}+\{11.21\}$ remains.

There is a choice of either $\{11.31\}$ or $\{11.4\}+\{11.2^2\}$. Obviously the latter alternative must be taken, as both terms represent reducible concomitants,

$$\{11.4\} \rightarrow \{11.3\}+\{10.4\},$$
$$\{11.2^2\} \rightarrow \{11.21\}+\{10.2^2\}.$$

The remaining partitions commencing with 10 are

$$2\{10.4\}+2\{10.31\}+\{10.2^2\}.$$

The reducible concomitants commencing with 10 are

$$\{10.41\} \rightarrow \{10.4\}+\{10.31\}+\{941\},$$
$$\{10.32\} \rightarrow \{10.31\}+\{10.2^2\}+\{932\},$$
$$\{10.5\} \rightarrow \{10.4\}+\{95\}.$$

These exhaust the 10's. The 9's give

$$\{95\}+2\{941\}+2\{932\}+\{92^21\}.$$

Reducible concomitants give

$$2\{942\} \rightarrow 2\{941\}+2\{932\}+2\{842\},$$
$$\{92^3\} \rightarrow \{92^21\}+\{82^3\},$$
$$\{96\} \rightarrow \{95\}+\{86\},$$

and these exhaust the 9's.

The 8's give $\{86\}+2\{851\}+3\{842\}+\{841^2\}+\{83^2\}+2\{8321\}+\{82^3\}$.

Reducible concomitants give

$$\{861\} \rightarrow \{86\}+\{851\}+\{761\},$$
$$\{8421\} \rightarrow \{842\}+\{841^2\}+\{8321\}+\{7421\}.$$

To use up $\{82^3\}$ then

$$\{832^2\} \to \{8321\}+\{82^3\}+\{732^2\},$$

and, to use $\{83^2\}$

$$\{843\} \to \{842\}+\{83^2\}+\{743\},$$

and lastly

$$\{852\} \to \{851\}+\{842\}+\{752\}.$$

The 7's give

$$\{761\}+2\{752\}+\{751^2\}+2\{743\}+3\{7421\}+\{73^21\}+\{732^2\}+\{72^31\}.$$

Commencing with the last term, the only possibility is

$$\{72^4\} \to \{72^31\}+\{62^4\}.$$

It is not possible to take $\{7332\}$ to use up $\{73^21\}+\{732^2\}$, as the three terms $\{7421\}$ could not then be successfully combined with other terms. Now linking

$$\{742^2\} \to \{7421\}+\{732^2\}+\{642^2\},$$
$$\{7431\} \to \{743\}+\{7421\}+\{73^21\}+\{6431\},$$
$$\{7521\} \to \{752\}+\{751^2\}+\{7421\}+\{6521\},$$
$$\{74^2\} \to \{743\}+\{64^2\},$$
$$\{762\} \to \{761\}+\{752\}+\{6^22\}.$$

The 6's give

$$\{6^22\}+\{653\}+\{6521\}+\{64^2\}+\{6431\}+2\{642^2\}+\{6421^2\}+\{632^21\}.$$

To use the first term it is necessary to take

$$\{6^23\} \to \{6^22\}+\{653\}.$$

To use $\{64^2\}$, since $\{654\}$ would require an extra $\{653\}$ which is not present, take

$$\{64^21\} \to \{64^2\}+\{6431\}+\{54^21\}.$$

The two terms $\{642^2\}$ then indicate that

$$\{6522\} \to \{6521\}+\{642^2\}+\{5^22^2\}$$
$$\{642^21\} \to \{642^2\}+\{6421^2\}+\{632^21\}+\{542^21\}.$$

There remain the terms

$$\{5^231\}+\{5^221^2\}+\{54^21\}+\{5432\}+\{5431^2\}+\{4^32\}.$$

The last term indicates definitely

$$\{54^22\} \to \{54^21\}+\{5432\}+\{4^32\},$$

and finally

$$\{5^231^2\} \to \{5^231\}+\{5^221^2\}+\{5431^2\}.$$

Thus the complete answer is

$$\begin{aligned}\{3\}\otimes\{5\} = {} & \{15\}+\{13\,.\,2\}+\{12\,.\,3\}+\{11\,.\,4\}+\{11\,.\,2^2\}+\{10\,.\,5\}+\{10\,.\,41\}+\{10\,.\,32\} \\ & +\{96\}+2\{942\}+\{92^3\}+\{861\}+\{852\}+\{843\}+\{8421\}+\{832^2\}+\{762\} \\ & +\{7521\}+\{74^2\}+\{7431\}+\{742^2\}+\{72^4\}+\{6^23\}+\{652^2\}+\{64^21\}+\{642^21\} \\ & +\{54^22\}+\{5^231^2\}.\end{aligned}$$

A similar analysis determines

$$\begin{aligned}\{3\}\otimes\{6\} = {} & \{18\}+\{16.2\}+\{15.3\}+\{14.4\}+\{14.2^2\}+\{13.5\}+\{13.41\}+\{13.32\}\\ & +2\{12.6\}+2\{12.42\}+\{12.2^3\}+\{11.61\}+2\{11.52\}+\{11.43\}+\{11.421\}\\ & +\{11.32^2\}+\{10.8\}+\{10.71\}+2\{10.62\}+\{10.53\}+\{10.521\}+2\{10.4^2\}\\ & +\{10.431\}+2\{10.42^2\}+\{10.2^4\}+\{972\}+2\{963\}+\{9621\}+2\{952^2\}+\{9531\}\\ & +2\{94^21\}+\{9432\}+\{942^21\}+\{932^3\}+\{8^22\}+\{8721\}+2\{864\}+\{8631\}\\ & +2\{862^2\}+\{8541\}+\{8532\}+\{8531^2\}+\{852^21\}+2\{84^22\}+\{84321\}+\{842^3\}\\ & +\{82^5\}+\{7^231\}+\{7641\}+\{7632\}+\{762^21\}+\{7542\}+\{7541^2\}+\{75321\}\\ & +\{742^31\}+\{74^221\}+\{74^23\}+\{6^3\}+\{6^242\}+\{6^22^3\}+\{64^3\}+\{65421\}\\ & +\{65321^2\}+\{64^22^2\}+\{5^2431\}+\{5^31^3\}.\end{aligned}$$

The types of the irreducible concomitants of a cubic up to degree 6 are as follows. In six or more variables all the concomitants exist. If there are less than six variables those partitions only are taken for which the number of parts does not exceed the number of variables.

Degree 2; $\{42\}$.

Degree 3; $\{63\}$, $\{52^2\}$, $\{4^21\}$.

Degree 4; $\{6^2\}$, $\{732\}$, $\{642\}$, $\{62^3\}$, $\{5421\}$, $\{4^3\}$.

Degree 5; $\{852\}$, $\{843\}$, $\{832^2\}$, $\{762\}$, $\{7521\}$, $\{7431\}$, $\{742^2\}$, $\{72^4\}$, $\{6^23\}$, $\{652^2\}$, $\{64^21\}$, $\{642^21\}$, $\{5^231^2\}$, $\{54^22\}$.

Degree 6; $\{10.53\}$, $\{972\}$, $\{952^2\}$, $\{9531\}$, $\{94^21\}$, $\{9432\}$, $\{932^3\}$, $\{8721\}$, $\{864\}$, $\{8631\}$, $2\{862^2\}$, $\{8541\}$, $\{8532\}$, $\{852^21\}$, $\{84^22\}$, $\{84321\}$, $\{842^3\}$, $\{82^5\}$, $\{7^231\}$, $\{7641\}$, $\{7632\}$, $\{762^21\}$, $\{7542\}$, $\{7541^2\}$, $\{75321\}$, $\{742^31\}$, $\{74^221\}$, $\{74^23\}$, $\{6^3\}$, $\{6^242\}$, $\{6^22^3\}$, $\{64^3\}$, $\{65421\}$, $\{65321^2\}$, $\{64^22^2\}$, $\{5^31^3\}$, $\{5^2431\}$.

There are two syzygies of degree 6 in the coefficients, which are of types $\{12.6\}$ and $\{11.52\}$ respectively.

The determination of the actual concomitants cannot yet be made with the same directness as the determination of the types. However, when the types are known the number of possible tableaux is strictly limited, and the method of trial and error does not involve very much labour.

It is sufficient to form sequences of partitions through the consecutive degrees. The tableaux may then be built by constructing tableaux corresponding to consecutive partitions. The sequences up to degree 5 are indicated.

The tableau for $\{42\}$ is unambiguous. Since this is the only irreducible concomitant of degree 2, the tableaux for each irreducible concomitant of degree 3, namely, those of type $\{63\}$, $\{52^2\}$, $\{4^21\}$, must be built from this.

Of degree 4, the concomitant $\{6^2\}$ is obviously built from $\{63\}$, the concomitant $\{62^3\}$ from $\{52^2\}$, and the concomitant $\{4^3\}$ from $\{4^21\}$. A tableau for the type $\{732\}$ could be built from either $\{63\}$ or $\{52^2\}$. Now consider the first case. The tableau would then be

$$\begin{pmatrix}\alpha\,\alpha\,\alpha\,\beta\,\gamma\,\gamma\,\delta\\ \beta\,\beta\,\gamma & & \\ \delta\,\delta & & \end{pmatrix}.$$

If the corresponding algebraic form is not zero it must be the required concomitant. Examination is made of the leading coefficient, i.e. the coefficient of $x_1^7 y_2^3 z_3^2$. Let the cubic be $a_{ijk} x^i x^j x^k$.

Consider the terms in the leading coefficient which contain the factor a_{332}. Symbolically, this can be either $\alpha_3^2 \alpha_2$, $\beta_3^2 \beta_2$, $\gamma_3^2 \gamma_2$ or $\delta_3^2 \delta_2$. Since, considering the last four columns, the factors β_1, γ_1, δ_1 are present in each expression, the only possibility is $\alpha_3^2 \alpha_2$. The co-factor is obtained from the tableau

$$-\begin{pmatrix} \beta & \beta & \gamma & \beta & \gamma & \gamma & \delta \\ \delta & \delta & & & & & \end{pmatrix}$$

which is not zero. The tableau represents the required concomitant.

For the concomitant $\{642\}$ the possibilities are $\{63\}$ or $\{52^2\}$. Taking the first alternative, then

$$\begin{pmatrix} \alpha & \alpha & \alpha & \beta & \gamma & \gamma \\ \beta & \beta & \gamma & \delta & & \\ \delta & \delta & & & & \end{pmatrix}.$$

The corresponding expression must be zero, however, for the interchange of β's and δ's changes the sign. From $\{52^2\}$ is obtained

$$\begin{pmatrix} \alpha & \alpha & \alpha & \beta & \gamma & \delta \\ \beta & \beta & \delta & \delta & & \\ \gamma & \gamma & & & & \end{pmatrix}.$$

Now pick out the coefficient of a_{332} in the leading coefficient. The coefficient of $\alpha_3 \alpha_3 \alpha_2$ is as obtained from the tableau

$$-\begin{pmatrix} \beta & \beta & \beta & \gamma & \delta & \delta \\ \gamma & \gamma & \delta & & & \end{pmatrix},$$

which is not zero.

The coefficient of $\beta_2 \beta_3 \beta_3$ is the same. These are the only possibilities, and thus the tableau represents the concomitant.

The concomitant $\{5421\}$ might be obtained from either $\{52^2\}$ or $\{4^21\}$. The latter alternative gives

$$\begin{pmatrix} \alpha & \alpha & \alpha & \beta & \delta \\ \beta & \beta & \gamma & \gamma & \\ \gamma & \delta & & & \\ \delta & & & & \end{pmatrix}.$$

We do not consider the coefficient of a_{432} in the leading coefficient, because the corresponding tableau would be of type $\{531\}$ and there is no such concomitant. Now consider a_{431}.

The coefficient of $\delta_4 \delta_3 \delta_1$ is obtained from the tableau for the concomitant $\{4^21\}$. The following tableau is obtained for the coefficient of $\alpha_4 \alpha_3 \alpha_1$:

$$-\begin{pmatrix} \beta & \beta & \;\; & \beta & \delta \\ \gamma & \delta & \gamma & \gamma & \\ \delta & & & & \end{pmatrix}.$$

Corresponding to the 3rd column γ_2 must be taken. Transferring the δ from the 5th to the 3rd column half the terms from the tableau must be taken:

$$-\begin{pmatrix}\beta\,\beta\,\delta\,\beta\\ \gamma\,\delta\,\gamma\,\gamma\\ \delta\end{pmatrix} \quad \text{or} \quad +\begin{pmatrix}\beta\,\beta\,\beta\,\gamma\\ \gamma\,\gamma\,\delta\,\delta\\ \delta\end{pmatrix}$$

which gives half of the concomitant $\{4^21\}$. The coefficient of $\beta_3^2\beta_2$ gives a similar result. Clearly the coefficient of a_{431} is not zero, and the tableau gives the required concomitant.

In a similar way the following sequences may be verified as giving correct tableaux for concomitants. The partition on the left is followed by the types of concomitant of degree one greater which are obtained from it:

$$\{42\};\ \{63\},\ \{4^21\},\ \{52^2\}.$$
$$\{63\};\ \{732\},\ \{6^2\}.$$
$$\{52^2\};\ \{642\},\ \{62^3\}.$$
$$\{4^21\};\ \{5421\},\ \{4^3\}.$$
$$\{732\};\ \{852\},\ \{832^2\},\ \{762\},\ \{7431\}.$$
$$\{642\};\ \{843\},\ \{742^2\},\ \{6^23\},\ \{7521\}.$$
$$\{62^3\};\ \{72^4\}.$$
$$\{4^3\};\ \{54^22\}.$$
$$\{5421\};\ \{7431\},\ \{652^2\},\ \{64^21\},\ \{5^231^2\}.$$

Thus corresponding to the partition $\{7521\}$ the sequence is obtained in reverse order:

$$\{7521\},\quad \{642\},\quad \{52^2\},\quad \{42\}.$$

Hence the tableau is

$$\begin{pmatrix}\alpha\,\alpha\,\alpha\,\beta\,\gamma\,\delta\,\epsilon\\ \beta\,\beta\,\delta\,\delta\,\epsilon\\ \gamma\,\gamma\\ \epsilon\end{pmatrix}.$$

The corresponding concomitant is

$$(\alpha\beta\gamma\epsilon;\ xyzw)\ (\alpha\beta\gamma;\ xyz)\ (\alpha\delta;\ xy)\ (\beta\delta;\ xy)\ (\gamma\epsilon;\ xy)\ \delta_x\epsilon_x.$$

The sequences define all concomitants of the cubic up to degree 5 in the coefficients.

The complete system for the ternary cubic has been obtained by Clebsch & Gordan (1875), but the above results in more than three variables are believed to be new.

The quartic

The concomitants are obtained of the quartic* in any number of variables up to degree 5 in the coefficients. The third method described above is used.

Thus $$\{4\}\{3\} = \{7\}+\{61\}+\{52\}+\{43\},$$

and
$$\{8\}\to\{7\},$$
$$\{62\}\to\{61\}+\{52\},$$
$$\{44\}\to\{43\}.$$

Hence $$\{4\}\otimes\{2\} = \{8\}+\{62\}+\{4^2\}.$$

* For the ternary quartic a notable but not complete account is given by Noether (1908).

Next
$$\begin{aligned}\{4\}\otimes\{2\}\{3\} &= [\{8\}+\{62\}+\{4^2\}]\,\{3\}\\ &= \{11\}+\{10\,.\,1\}+2\{92\}+2\{83\}+\{821\}+2\{74\}+\{731\}\\ &\qquad +\{72^2\}+\{65\}+2\{641\}+\{632\}+\{542\}+\{4^23\},\end{aligned}$$

and
$$\begin{aligned}\{12\}&\rightarrow\{11\},\\ \{10\,.\,2\}&\rightarrow\{10\,.\,1\}+\{92\},\\ \{93\}&\rightarrow\{92\}+\{83\},\\ \{84\}&\rightarrow\{83\}+\{74\},\\ \{82^2\}&\rightarrow\{821\}+\{72^2\},\\ \{741\}&\rightarrow\{74\}+\{731\}+\{641\},\\ \{642\}&\rightarrow\{641\}+\{632\}+\{542\},\\ \{4^3\}&\rightarrow\{4^23\}.\end{aligned}$$

Hence
$$\{4\}\otimes\{3\}=\{12\}+\{10\,.\,2\}+\{93\}+\{84\}+\{82^2\}+\{741\}+\{642\}+\{4^3\}+\{6^2\}.$$

For $\{4\}\otimes\{3\}\{3\}$ then
$$\begin{aligned}&\{15\}+\{14\,.\,1\}+2\{13\,.\,2\}+3\{12\,.\,3\}+\{12\,.\,21\}+3\{11\,.\,4\}+2\{11\,.\,31\}+2\{11\,.\,2^2\}\\ &\quad+3\{10\,.\,5\}+4\{10\,.\,41\}+3\{10\,.\,32\}+\{10\,.\,2^21\}+3\{96\}+3\{951\}+5\{942\}+\{941^2\}\\ &\quad+\{9321\}+\{92^21\}+\{87\}+3\{861\}+4\{852\}+3\{843\}+3\{8421\}+\{832^2\}+\{7^21\}\\ &\quad+3\{762\}+\{761^2\}+2\{753\}+2\{7521\}+3\{744\}+2\{7431\}+\{742^2\}+2\{6^23\}+\{6^221\}\\ &\quad+\{654\}+\{6531\}+\{652^2\}+2\{64^21\}+\{6432\}+\{54^22\}+\{4^33\}.\end{aligned}$$

From these
$$\begin{aligned}\{4\}\otimes\{4\}&=\{16\}+\{14\,.\,2\}+\{13\,.\,3\}+2\{12\,.\,4\}+\{12\,.\,2^2\}+\{11\,.\,41\}+\{11\,.\,32\}+2\{10\,.\,6\}\\ &\quad+\{10\,.\,51\}+2\{10\,.\,42\}+\{10\,.\,2^3\}+\{961\}+\{952\}+\{943\}+\{9421\}+\{8^2\}\\ &\quad+2\{862\}+\{842^2\}+\{8521\}+2\{84^2\}+\{7^21^2\}+\{763\}+\{7531\}+\{74^21\}+\{6^24\}\\ &\quad+\{6^22^2\}+\{64^22\}+\{4^4\}.\end{aligned}$$

In a similar manner
$$\begin{aligned}\{4\}\otimes\{5\}&=\{20\}+\{18\,.\,2\}+\{17\,.\,3\}+2\{16\,.\,4\}+\{16\,.\,2^2\}+\{15\,.\,5\}+\{15\,.\,41\}+\{15\,.\,32\}\\ &\quad+2\{14\,.\,6\}+\{14\,.\,51\}+3\{14\,.\,42\}+\{14\,.\,2^3\}+\{13\,.\,7\}+2\{13\,.\,61\}+2\{13\,.\,52\}\\ &\quad+2\{13\,.\,43\}+\{13\,.\,421\}+\{13\,.\,32^2\}+2\{12\,.\,8\}+\{12\,.\,71\}+4\{12\,.\,62\}+\{12\,.\,53\}\\ &\quad+2\{12\,.\,521\}+3\{12\,.\,4^2\}+\{12\,.\,431\}+2\{12\,.\,42^2\}+\{12\,.\,2^4\}+\{11\,.\,81\}\\ &\quad+2\{11\,.\,72\}+\{11\,.\,71^2\}+3\{11\,.\,63\}+2\{11\,.\,621\}+\{11\,.\,54\}+2\{11\,.\,531\}\\ &\quad+2\{11\,.\,52^2\}+2\{11\,.\,4^21\}+\{11\,.\,432\}+\{11\,.\,42^21\}+\{10^2\}+2\{10\,.\,82\}+2\{10\,.\,73\}\\ &\quad+2\{10\,.\,721\}+4\{10\,.\,64\}+2\{10\,.\,631\}+3\{10\,.\,62^2\}+2\{10\,.\,541\}+\{10\,.\,532\}\\ &\quad+\{10\,.\,531^2\}+\{10\,.\,52^21\}+3\{10\,.\,4^22\}+\{10\,.\,42^3\}+\{983\}+\{9821\}+\{974\}\\ &\quad+2\{9731\}+\{972^2\}+\{9721^2\}+\{965\}+3\{9641\}+2\{9632\}+\{962^21\}+2\{9542\}\\ &\quad+\{9541^2\}+\{95321\}+\{94^23\}+\{94^221\}+2\{84^3\}+\{84^22^2\}+\{86321\}+\{862^3\}\\ &\quad+2\{86^2\}+\{8651\}+3\{8642\}+2\{8741\}+\{8543\}+\{85421\}+\{8732\}+\{85^21^2\}\\ &\quad+\{8731^2\}+2\{8^24\}+2\{8^22^2\}+\{7^241^2\}+\{7^251\}+\{7^23^2\}+\{75431\}+\{74^31\}\\ &\quad+\{7652\}+\{76421\}+\{7643\}+\{6^24^2\}+\{6^242^2\}+\{6^32\}+\{64^32\}+\{4^5\}.\end{aligned}$$

The irreducible concomitants are of type

Degree 2; $\{62\}$, $\{4^2\}$.

Degree 3; $\{93\}$, $\{82^2\}$, $\{741\}$, $\{6^2\}$, $\{642\}$, $\{4^3\}$.

Degree 4; $\{11.32\}$, $\{10.51\}$, $\{10.42\}$, $\{10.2^3\}$, $\{961\}$, $\{952\}$, $\{943\}$, $\{9421\}$, $2\{862\}$, $\{842^2\}$, $\{8521\}$, $\{84^2\}$, $\{7^21^2\}$, $\{763\}$, $\{7531\}$, $\{74^21\}$, $\{6^24\}$, $\{6^22^2\}$, $\{64^22\}$, $\{4^4\}$.

Degree 5; $\{13.52\}$, $\{13.43\}$, $\{13.32^2\}$, $\{12.71\}$, $\{12.53\}$, $\{12.521\}$, $\{12.4^2\}$, $\{12.431\}$, $\{12.42^2\}$, $\{12.2^4\}$, $2\{11.72\}$, $2\{11.63\}$, $2\{11.621\}$, $\{11.54\}$, $\{11.531\}$, $2\{11.52^2\}$, $\{11.4^21\}$, $\{11.432\}$, $\{11.42^21\}$, $\{10.82\}$, $2\{10.73\}$, $2\{10.721\}$, $2\{10.64\}$, $2\{10.631\}$, $2\{10.62^2\}$, $2\{10.541\}$, $\{10.532\}$, $\{10.531^2\}$, $\{10.52^21\}$, $2\{10.4^22\}$, $\{10.42^3\}$, $\{983\}$, $\{9821\}$, $\{974\}$, $2\{9731\}$, $\{972^2\}$, $\{9721^2\}$, $\{965\}$, $3\{9641\}$, $2\{9632\}$, $\{962^21\}$, $\{84^3\}$, $\{84^22^2\}$, $\{86321\}$, $\{862^3\}$, $2\{86^2\}$, $\{8651\}$, $3\{8642\}$, $2\{8741\}$, $\{8543\}$, $\{85421\}$, $\{8732\}$, $\{85^21^2\}$, $\{8731^2\}$, $\{8^24\}$, $2\{8^22^2\}$, $\{75431\}$, $\{74^31\}$, $\{7652\}$, $\{76421\}$, $\{7643\}$, $\{7^241^2\}$, $\{7^251\}$, $\{7^23^2\}$, $\{6^24^2\}$, $\{6^242^2\}$, $\{6^32\}$, $\{64^32\}$, $\{4^5\}$.

The sequences which enable the concomitants up to degree 4 to be constructed are

$\{62\}$; $\{93\}$, $\{82^2\}$, $\{6^2\}$, $\{642\}$, $\{741\}$.

$\{4^2\}$; $\{4^3\}$.

$\{93\}$; $\{11.32\}$, $\{10.51\}$, $\{952\}$.

$\{82^2\}$; $\{10.42\}$, $\{10.2^3\}$, $\{9421\}$.

$\{741\}$; $\{961\}$, $\{943\}$, $\{7^21^2\}$, $\{862\}$.

$\{6^2\}$; $\{862\}$, $\{6^24\}$.

$\{642\}$; $\{842^2\}$, $\{84^2\}$, $\{6^22^2\}$, $\{64^22\}$, $\{763\}$, $\{8521\}$, $\{7531\}$, $\{74^21\}$.

$\{4^3\}$; $\{4^4\}$.

These sequences are not always unique, e.g. a concomitant corresponding to $\{862\}$ could be obtained from either $\{82^2\}$ or $\{642\}$, but these concomitants would be the same as that obtained from $\{6^2\}$.

The quadratic complex

The types of concomitants up to degree 4 in the coefficients of a quadratic complex in any number of variables are now obtained.

Using the third method described above it is seen that, if

$$\{2^2\}\otimes\{2\} = \Sigma\{\lambda\},$$

then

$$\{2^2\}\{21\} = \Sigma g_{1\mu\lambda}\{\mu\} = \{43\}+\{421\}+\{3^21\}+\{32^2\}+\{321^2\}+\{2^31\}.$$

Starting with the reducible concomitant

$$\{4^2\}\rightarrow\{43\},$$

the only possibilities are then

$$\{42^2\}\rightarrow\{421\}+\{32^2\},$$

$$\{3^21^2\}\rightarrow\{3^21\}+\{321^2\},$$

$$\{2^4\}\rightarrow\{2^31\}.$$

Thus

$$\{2^2\}\otimes\{2\} = \{4^2\}+\{42^2\}+\{3^21^2\}+\{2^4\}.$$

Evaluating $\{2^2\}\otimes\{3\}$ it is found that

$$\begin{aligned}\{2^2\}\otimes\{2\}\{21\} = {} & \{65\}+\{641\}+\{5^21\}+\{542\}+\{541^2\}+\{4^221\}+\{632\}+\{62^21\}+\{542\}\\ & +\{53^2\}+2\{5321\}+\{52^3\}+\{52^21^2\}+\{4^23\}+\{4^221\}+\{43^21\}+\{432^2\}\\ & +\{4321^2\}+\{42^31\}+\{541^2\}+\{5321\}+\{531^3\}+\{4^221\}+\{4^21^3\}+\{4331\}\\ & +\{432^2\}+2\{4321^2\}+\{431^4\}+\{3^32\}+\{3^31^2\}+\{3^22^21\}+\{3^221^3\}+\{432^2\}\\ & +\{42^31\}+\{3^22^21\}+\{32^4\}+\{32^31^2\}+\{2^51\}.\end{aligned}$$

The reducible concomitants give

$$\begin{aligned}\{6^2\}&\to\{65\},\\ \{642\}&\to\{641\}+\{632\}+\{542\},\\ \{5^21^2\}&\to\{5^21\}+\{541^2\},\\ \{4^22^2\}&\to\{4^221\}+\{432^2\}.\end{aligned}$$

To utilize the other S-functions, take

$$\begin{aligned}\{62^3\}&\to\{62^21\}+\{52^3\},\\ \{53^21\}&\to\{53^2\}+\{5321\}+\{43^21\},\\ \{5421\}&\to\{542\}+\{541^2\}+\{5321\}+\{4^221\},\\ \{5321^2\}&\to\{5321\}+\{531^3\}+\{52^21^2\}+\{4321^2\},\\ \{4^3\}&\to\{4^23\},\\ \{4^22^2\}&\to\{4^221\}+\{432^2\},\\ \{4^21^4\}&\to\{4^21^3\}+\{431^4\},\\ \{43^21^2\}&\to\{43^21\}+\{4321^2\}+\{3^31^2\},\\ \{432^21\}&\to\{432^2\}+\{4321^2\}+\{42^31\}+\{3^22^21\},\\ \{42^4\}&\to\{42^31\}+\{32^4\},\\ \{3^4\}&\to\{3^32\},\\ \{3^22^21^2\}&\to\{3^22^21\}+\{3^221^3\}+\{32^31^2\},\\ \{2^6\}&\to\{2^51\}.\end{aligned}$$

Hence

$$\begin{aligned}\{2^2\}\otimes\{3\} = {} & \{6^2\}+\{642\}+\{62^3\}+\{5^21^2\}+\{5421\}+\{53^21\}+\{5321^2\}+\{4^3\}+2\{4^22^2\}\\ & +\{4^21^4\}+\{43^21^2\}+\{432^21\}+\{42^4\}+\{3^4\}+\{3^22^21^2\}+\{2^6\}.\end{aligned}$$

To evaluate $\{2^2\}\otimes\{4\}$ it is easier to use the second method. Using the same nomenclature as previously

$$\{1\}' = \{1\}+x,$$

and
$$\{2^2\}' = x^2\{2\}+x\{21\}+\{2^2\}.$$

Then $$\{2^2\}'\otimes\{4\} = x^8\{2\}\otimes\{4\}+x^7\{2\}\otimes\{3\}\{21\}+x^6[\{2\}\otimes\{3\}\{2^2\}+\{2\}\otimes\{2\}\{21\}\otimes\{2\}]+\ldots.$$

The coefficient of x^8 is

$$\{2\}\otimes\{4\} = \{8\}+\{62\}+\{4^2\}+\{42^2\}+\{2^4\}.$$

The corresponding terms in $\{2^2\}' \otimes \{4\}$ are

$$\{8^2\}+\{862\}+\{84^2\}+\{842^2\}+\{82^4\}.$$

The coefficient of x^7 is

$$\begin{aligned}\{2\}\otimes\{3\}\{21\} &= [\{6\}+\{42\}+\{2^3\}]\,\{21\}\\ &= \{81\}+\{72\}+\{71^2\}+\{621\}+\{63\}+\{621\}+\{54\}+\{531\}+\{521^2\}\\ &\quad+\{531\}+\{52^2\}+\{4^21\}+\{432\}+\{431^2\}+\{42^21\}+\{432\}+\{42^21\}\\ &\quad+\{3^221\}+\{32^3\}+\{32^21^2\}+\{2^41\}.\end{aligned}$$

The S-functions already obtained account for

$$\{81\}+\{72\}+\{63\}+\{621\}+\{54\}+\{4^21\}+\{52^2\}+\{432\}+\{42^21\}+\{32^3\}+\{2^41\}.$$

The remaining terms give the S-functions with 7 as the initial part, namely

$$\{7^21^2\}'+\{7621\}'+2\{7531\}'+\{7321^2\}'+\{7432\}'+\{7431^2\}'+\{742^21\}'+\{73^221\}'+\{732^21^2\}'.$$

The coefficient of x^6 is

$$\begin{aligned}&\{2\}\otimes\{3\}\{2^2\}+\{2\}\otimes\{2\}\{21\}\otimes\{2\}\\ &= [\{6\}+\{42\}+\{2^3\}]\,\{2^2\}+[\{4\}+\{2^2\}]\,[\{42\}+\{321\}+\{2^3\}+\{31^3\}]\\ &= \{82\}+\{73\}+\{721\}+\{64\}+\{631\}+\{622\}+\{541\}+\{532\}+\{4^22\}+\{721\}+\{631\}\\ &\quad+\{62^2\}+\{621^2\}+\{532\}+\{531^2\}+\{52^21\}+\{4321\}+\{62^2\}+\{52^21\}+\{42^3\}+\{71^3\}\\ &\quad+\{621^2\}+\{61^4\}+\{531^2\}+\{521^3\}+\{431^3\}+\{64\}+\{631\}+\{62^2\}+\{541\}+\{532\}\\ &\quad+\{531^2\}+\{52^21\}+\{4^22\}+\{4321\}+\{42^31\}+\{541\}+\{532\}+\{531^2\}+\{52^21\}+\{4^22\}\\ &\quad+\{4^21^2\}+\{43^2\}+\{4321\}+\{4321\}+\{42^3\}+\{42^21^2\}+\{4^22\}+\{431^2\}+\{42^3\}+\{3^221^2\}\\ &\quad+\{32^31\}+\{2^5\}+\{32^31\}+\{3^221^2\}+\{3^31\}+\{531^2\}+\{521^3\}+\{4321\}+\{431^3\}+\{42^21^2\}\\ &\quad+\{421^4\}+\{3^31\}+\{3^221^2\}+\{32^21^3\}+\{82\}+\{721\}+\{62^2\}+\{64\}+\{631\}+\{62^2\}+\{541\}\\ &\quad+\{532\}+\{531^2\}+\{52^21\}+\{4^22\}+\{4321\}+\{42^3\}+\{4^22\}+\{4321\}+\{42^3\}+\{3^221^2\}\\ &\quad+\{32^31\}+\{2^5\}.\end{aligned}$$

Of these the known S-functions account for

$$\begin{aligned}&\{82\}+\{73\}+\{721\}+\{64\}+\{631\}+\{62^2\}+\{541\}+\{532\}+\{4^22\}+\{721\}+\{631\}+\{62^2\}\\ &\quad+\{621^2\}+\{532\}+\{531^2\}+\{52^21\}+\{52^21\}+\{71^3\}+\{531^2\}+\{631\}+\{62^2\}+\{4^22\}\\ &\quad+\{4321\}+\{42^3\}+\{541\}+\{532\}+\{531^2\}+\{4^21^2\}+\{43^2\}+\{4321\}+\{42^21^2\}+\{4321\}\\ &\quad+\{42^3\}+\{32^31\}+\{4321\}+\{431^3\}+\{42^21^2\}+\{3^31\}+\{3^221^2\}+\{32^21^3\}+\{82\}+\{721\}\\ &\quad+\{64\}+\{631\}+\{541\}+\{532\}+\{4^22\}+\{4321\}+\{42^3\}+\{3^221^2\}+\{32^31\}+\{2^5\}.\end{aligned}$$

The remaining terms give the S-functions with initial part 6, namely,

$$\begin{aligned}&\{6^21^4\}+\{6^24\}+3\{6^22^2\}+\{6521^3\}+\{6541\}+\{6532\}+2\{6531^2\}+2\{652^21\}+3\{64^22\}\\ &\quad+3\{64321\}+3\{642^3\}+2\{6431^3\}+\{64214\}+\{63^31\}+2\{63^221^2\}+\{632^31\}+\{62^5\}.\end{aligned}$$

Now by reason of the theorem of conjugates which will be proved later, since $\{2^2\}$ is a self-conjugate partition of an even number, for every S-function in $\{2^2\}\otimes\{4\}$ the S-function corresponding to the conjugate partition will also appear. Thus another 31 terms are obtained in $\{2^2\}\otimes\{4\}$. There remain only the partitions with $\leqslant 5$ parts in which each part $\leqslant 5$. These are easily obtained by the third method described above which also provides a check on the terms already obtained. If

$$\{2^2\}\otimes\{4\} = \Sigma\{\lambda\},$$

then

$$\{2^2\}\otimes\{3\}\{21\} = \Sigma g_{1\mu\lambda}\{\mu\}.$$

Thus

$$\begin{aligned}\{2^2\}\otimes\{4\} = {} & \{8^2\}+\{862\}+\{84^2\}+\{842^2\}+\{82^4\}+\{7^21^2\}+\{7621\}+2\{7531\}+\{7432\} \\ & +\{7431^2\}+\{742^21\}+\{73^221\}+\{732^21^2\}+\{6^24\}+3\{6^22^2\}+\{6^21^4\}+\{6541\} \\ & +\{6532\}+\{6521^3\}+2\{6531^2\}+2\{652^21\}+3\{64^22\}+3\{64321\}+3\{642^3\} \\ & +2\{6431^3\}+\{6421^4\}+\{63^31\}+2\{6321^2\}+\{632^31\}+\{62^5\}+\{5^21^6\} \\ & +\{53^221^3\}+\{542^21^3\}+\{532^31^2\}+\{5431^4\}+\{542^31\}+2\{53^22^21\}+\{542^31\} \\ & +2\{5^22^21^2\}+3\{54321^2\}+\{54^21^3\}+3\{5^23^2\}+2\{5^241^2\}+2\{5^2321\}+2\{543^21\} \\ & +2\{54^221\}+2\{5432^2\}+2\{53^32\}+2\{4^3\}+3\{4^32^2\}+3\{4^23^21^2\}+\{4^2321\} \\ & +\{43^321\}+3\{4^22^2\}+\{4^2321^3\}+2\{43^22^21^2\}+\{432^41\}+\{42^6\}+\{4^22^21^4\} \\ & +\{3^42^2\}+\{3^41^6\}+\{3^22^41^2\}+\{2^8\}.\end{aligned}$$

The theorem of conjugates

The types of concomitant of a quadratic and those of a linear complex show a notable resemblance. Thus there is a concomitant of a quadratic corresponding to every partition into parts of even magnitude only; there is a concomitant of a linear complex corresponding to every partition in which each part is repeated an even number of times.

This resemblance breaks down completely for the cubic and the conjugate form of type $\{1^3\}$. The concomitants of degree 2 of a cubic are of type $\{6\}$ and $\{42\}$; of a form of type $\{1^3\}$, they are of type $\{2^3\}$ and $\{21^4\}$. But the resemblance reappears for the quartic and the conjugate form of type $\{1^4\}$. Of degree 2, the types of concomitant of a quartic are $\{8\}$, $\{62\}$, $\{4^2\}$, and the types of concomitant of a form of type $\{1^4\}$ are $\{1^8\}$, $\{2^21^4\}$, $\{2^4\}$. The resemblance continues for higher degrees in the coefficients.

These resemblances prove to be particular cases of a very remarkable *theorem of conjugates* which proves incidentally that for each concomitant of type $\{\mu\}$ of a ground form of type $\{\lambda\}$, there is a concomitant of type $\{\tilde{\mu}\}$ of a ground form of type $\{\tilde{\lambda}\}$, provided that $\{\lambda\}$ is a partition of an even number. The sign $\sim$ denotes that the conjugate partition is taken.

The full theorem is more general than this, and is as follows:

The theorem of conjugates:

Let (λ) *be a partition of* p, *and let*

$$\{\lambda\}\otimes\{\mu\} = \Sigma\{\nu\}.$$

Then if p *is even*

$$\{\tilde{\lambda}\}\otimes\{\mu\} = \Sigma\{\tilde{\nu}\}$$

and if p *is odd*

$$\{\tilde{\lambda}\}\otimes\{\tilde{\mu}\} = \Sigma\{\tilde{\nu}\}.$$

The proof is as follows. Let S_r denote the sum of the rth powers of the characteristic roots of a matrix A, and let Z_r denote the sum of the rth powers of the characteristic roots of $A^{\{\lambda\}}$.

Then Z_1 is the spur of $A^{\{\lambda\}}$ which is

$$Z_1 = \{\lambda\} = \frac{1}{p!} \Sigma \chi_\rho^{(\lambda)} S_\rho,$$

where $\chi_\rho^{(\lambda)}$ is the characteristic corresponding to (λ) of the class $\rho = (1^a 2^b 3^c \ldots)$ of the symmetric group of order $p!$, and $S_\rho = S_1^a S_2^b S_3^c \ldots$ (Littlewood 1940, p. 86).

Now the characteristic roots of $A^{\{\lambda\}}$ are products of powers of the characteristic roots of A, and consequently Z_r can be obtained from the expression for Z_1 by replacing each characteristic root of A by its rth power, which implies that S_p must be replaced by S_{pr}. Thus

$$Z_r = \frac{1}{p!} \Sigma \chi_\rho^{(\lambda)} S_r^a S_{2r}^b S_{3r}^c \ldots.$$

If (μ) is a partition of q, and ρ' denotes the class $(1^{a'} 2^{b'} 3^{c'} \ldots)$ of the symmetric group of order $q!$, then

$$\{\lambda\} \otimes \{\mu\} = \Sigma\{\nu\} = \frac{1}{q!} \Sigma \chi_{\rho'}^{(\mu)} Z_1^{a'} Z_2^{b'} Z_3^{c'} \ldots.$$

To obtain $\Sigma\{\tilde{\nu}\}$ from this expression each S_r must be replaced by $(-1)^{r+1} S_r$. Thus

$$Z_1 = \frac{1}{p!} \Sigma \chi_\rho^{(\lambda)} S_1^a S_2^b S_3^c \ldots$$

is replaced by
$$\frac{1}{p!} \Sigma (-1)^{b+d+\ldots} \chi_\rho^{(\lambda)} S_1^a S_2^b S_3^c \ldots = T_1,$$

where T_r denotes the sum of the rth powers of the characteristic roots of $A^{\{\tilde{\lambda}\}}$.

Also
$$Z_2 = \frac{1}{p!} \Sigma \chi_\rho^{(\lambda)} S_2^a S_4^b S_6^c \ldots$$

is replaced by
$$\frac{1}{p!} \Sigma (-1)^{a+b+c+\ldots} \chi_\rho^{(\lambda)} S_2^a S_4^b S_6^c \ldots.$$

Since
$$a + 2b + 3c + 4d + \ldots = p,$$

then
$$(-1)^{a+c+e+\ldots} = (-1)^p.$$

Hence Z_2 is replaced by

$$\frac{1}{p!} (-1)^p \Sigma (-1)^{b+d+\ldots} \chi_\rho^{(\lambda)} S_2^a S_4^b S_6^c \ldots = (-1)^p T_2.$$

Similarly Z_{2r+1} is replaced by T_{2r+1} and Z_{2r} by $(-1)^p T_{2r}$.

If p is even the same result is obtained if $\{\lambda\}$ is replaced by $\{\tilde{\lambda}\}$, i.e.

$$\{\tilde{\lambda}\} \otimes \{\mu\} = \Sigma\{\tilde{\nu}\},$$

but if p is odd allowance must be made for the factor $(-1)^p$ by replacing $\{\mu\}$ by $\{\tilde{\mu}\}$, i.e.

$$\{\tilde{\lambda}\} \otimes \{\tilde{\mu}\} = \Sigma\{\tilde{\nu}\}.$$

This theorem has a direct application to the concomitants of a single ground form only if (λ) is the partition of an even number, for if

$$\{\lambda\} \otimes \{n\} = \Sigma\{\nu\},$$

then
$$\{\tilde{\lambda}\} \otimes \{n\} = \Sigma\{\tilde{\nu}\}.$$

Thus since

$$\{4\}\otimes\{3\} = \{12\}+\{10.2\}+\{93\}+\{84\}+\{82^2\}+\{741\}+\{6^2\}+\{642\}+\{4^3\}$$

it is deduced that

$$\{1^4\}\otimes\{3\} = \{1^{12}\}+\{2^21^8\}+\{2^31^6\}+\{2^41^4\}+\{3^21^6\}+\{32^31^3\}+\{2^6\}+\{3^22^21^2\}+\{3^4\},$$

and from the concomitant of a quartic may be deduced the concomitant of a ground form of type $\{1^4\}$.

By an application of Young's methods of substitutional analysis, replacing symmetric groups by negative symmetric groups and vice versa, it may be shown that the interchange of rows and columns in any tableau giving a concomitant of a ground form of type $\{\lambda\}$ gives the correct tableau for the corresponding concomitant of a ground form of type $\{\tilde{\lambda}\}$. Thus the actual concomitant of a ground form of type $\{1^4\}$ may be deduced from those of the quartic.

When (λ) is the partition of an odd number, e.g. for the cubic, the method breaks down. The set of conjugate S-functions of the expansion of $\{3\}\otimes\{n\}$ is not $\{1^3\}\otimes\{n\}$ but $\{1^3\}\otimes\{1^n\}$. This gives, not the concomitant of a single ground form of type $\{1^3\}$, but the *alternating concomitant types* of a set of ground forms of type $\{1^3\}$. These will be discussed in the next section.

Returning to the case when (λ) is the partition of an even number, so that a correspondence is obtained between the concomitant of two ground forms of type $\{\lambda\}$ and $\{\tilde{\lambda}\}$ respectively, two points should be noticed.

First, the theorem deals with concomitants in any number of variables. Thus suppose that the concomitants of a ground form of type $\{31\}$ are known in five variables only. Then taking conjugates all the concomitants of a ground form of type $\{21^2\}$ are not obtained, but only those for which the first part does not exceed 5.

Secondly, it is the complete set of concomitants of a given degree that is involved, and not the irreducible set. The conjugate of a reducible concomitant may be irreducible, and vice versa. Thus

$$\{4\}\otimes\{2\} = \{8\}+\{62\}+\{4^2\}, \quad \{1^4\}\otimes\{2\} = \{1^8\}+\{2^21^4\}+\{2^4\}.$$

In the first case it is the concomitant of type $\{8\}$ which is reducible; in the second case, not $\{1^8\}$ but $\{2^4\}$ gives the reducible concomitant.

This difference in reducibility is a distinct aid in the determination of concomitants, for if $\{\lambda\}$ is the partition of an even number, in determining $\{\lambda\}\otimes\{n\}$, not only those terms corresponding to reducible concomitants can be written down immediately, but also those terms which would be reducible if the conjugates were taken.

Thus, in determining $\{4\}\otimes\{5\}$, the following terms could be written down immediately as they correspond to reducible terms in $\{1^4\}\otimes\{5\}$:

$$\{13.43\}, \{11.432\}, \{10.541\}, \{10.42^3\}, \{9641\}, \{9542\}, \{94^23\}, \{94^221\}, \{8642\},$$
$$\{84^22^2\}, \{85421\}, \{7^241^2\}, \{7643\}, \{75431\}, \{74^31\}, \{6^24^2\}, \{6^242^2\}, \{64^32\}, \{4^5\}.$$

This suggests a new problem. We consider a single ground form of type $\{\lambda\}$ where (λ) is a partition of an even number. It is known that the irreducible concomitants in a given number of variables are finite in number, but if the number of variables is made unlimited the number of concomitants becomes infinite, as is obvious even for the simple case of the quadratic. But if the definition of reducibility is revised so that a concomitant is considered reducible either if it is reducible in the old sense, or if the conjugate concomitant of the conjugate ground form is reducible, will the number of irreducible concomitants then become finite?

The answer is yes for the quadratic and for the linear complex. The ground form is then the only irreducible concomitant.

For the next simplest ground form, the quartic, the answer is, no. Thus the following infinite sequence of irreducible concomitants still exist:

$$\{4\},\quad \{62\},\quad \{82^2\},\quad \{10\,.\,2^3\},\quad \{12\,.\,2^4\},\quad \{14\,.\,2^6\},\quad \ldots.$$

Alternating concomitant types

The concomitant types, that is, the concomitants which are linear in the coefficients of each of p ground forms, each of type $\{\lambda\}$, are much more numerous than the concomitants of degree p in a single ground form of type $\{\lambda\}$.

Sometimes it is known, however, that a certain concomitant linear in each of p ground forms each of type $\{\lambda\}$ is such that each ground form plays an equivalent role, and the ground forms are not grouped together in any way. Such a concomitant must either be symmetric in the ground forms, or else antisymmetric. Since it is the vanishing of the concomitant which is usually considered, a change in sign of the concomitant when two ground forms are interchanged is not significant, and the antisymmetric case must be considered.

A concomitant which is symmetric in the p ground forms is of the same type as a concomitant of degree p in a single ground form, to which it becomes equal when the ground forms are made identical.

The remaining case is of some interest, namely, the case of concomitant types which are antisymmetric in the ground forms. These are called *alternating concomitant types*.

The alternating concomitant types of a set of p ground forms, each of type $\{\lambda\}$, correspond to the terms in the expansion of

$$\{\lambda\}\otimes\{1^p\}.$$

For quadratics use is made of a certain generating function (Littlewood 1940, p. 238 (11·9; 3)) to obtain the alternating concomitant types.

Let $\{\lambda\}$ denote an S-function of the quantities $\alpha, \beta, \gamma, \delta, \ldots$. Then

$$\{2\} = \Sigma\alpha^2 + \Sigma\alpha\beta,$$

and thus $\{2\}\otimes\{1^p\}$ is the sum of products of p different terms taken from the right-hand side of this equation.

Thus $\{2\}\otimes\{1^p\}$ is the coefficient of ρ^p in the expansion of

$$\Pi(1-\alpha^2\rho)\,\Pi(1-\alpha\beta\rho).$$

This is equal to (Littlewood 1940, p. 238 (11·9; 3))

$$1+\Sigma\{\gamma\}(-\rho)^w,$$

where (γ) is a partition of $2w$, and the summation is with respect to all partitions (γ) which in Frobenius's nomenclature is in one of the forms

$$\begin{pmatrix} r+1 \\ r \end{pmatrix}, \quad \begin{pmatrix} r+1, & s+1 \\ r, & s \end{pmatrix}, \quad \begin{pmatrix} r+1, & s+1, & t+1 \\ r, & s, & t \end{pmatrix}, \quad \dots.$$

Thus

$$\Pi(1-\alpha^2\rho)\,\Pi(1-\alpha\beta\rho) = 1-\{2\}\rho+\{31\}\rho^2-[\{41^2\}+\{3^2\}]\rho^3+[\{51^3\}+\{431\}]\rho^4 \\ -[\{61^4\}+\{531^2\}+\{4^22\}]\rho^5+\dots.$$

Hence

THEOREM. *There is an alternating concomitant type of quadratics corresponding to every partition which in Frobenius's nomenclature is in one of the forms*

$$\begin{pmatrix} r+1 \\ r \end{pmatrix}, \quad \begin{pmatrix} r+1, & s+1 \\ r, & s \end{pmatrix}, \quad \begin{pmatrix} r+1, & s+1, & t+1 \\ r, & s, & t \end{pmatrix}, \quad \dots.$$

The theorem of conjugates enables us to deduce immediately the theorem for linear complexes.

THEOREM. *There is an alternating concomitant type for linear complexes corresponding to every partition which in Frobenius's nomenclature is in one of the forms*

$$\begin{pmatrix} r \\ r+1 \end{pmatrix}, \quad \begin{pmatrix} r, & s \\ r+1, & s+1 \end{pmatrix}, \quad \begin{pmatrix} r, & s, & t \\ r+1, & s+1, & t+1 \end{pmatrix}, \quad \dots.$$

The actual concomitants can be obtained by building tableaux for consecutive degrees in the coefficients, such that each tableau is of the required type. Thus for the alternating invariant of 10 quaternary quadrics the tableau is

$$\begin{pmatrix} \alpha\ \alpha\ \beta\ \gamma\ \delta \\ \beta\ \xi\ \xi\ \eta\ \zeta \\ \gamma\ \eta\ \lambda\ \lambda\ \mu \\ \delta\ \zeta\ \mu\ \nu\ \nu \end{pmatrix}.$$

The product of five determinants corresponding to this tableau gives only one term in the required invariant. With concomitants of a single ground form one term is sufficient, for the fact that the ground forms are made identical is equivalent to the operation of the symmetric group of permutations on the ground forms. If instead of a concomitant of degree n in one ground form we sought a concomitant which was linear in each of n ground forms and symmetric in these ground forms, it would be necessary to operate with the symmetric group of permutations on these ground forms.

Similarly, to obtain the alternating invariant of 10 quaternary quadrics we must operate on the term given above with the negative symmetric group on the 10 quadrics.

Actually the 10! terms so obtained are not all required, as the given term already has the alternating property with respect to many permutations.

Turnbull & Young (1926) have shown that the alternating invariant of 10 quadrics can be expressed as the sum, with alternating signs, of 240 terms similar to that given above and obtainable from it by permutations of the quadrics.

The alternating concomitant types for the general ground form of type $\{\lambda\}$ may be found by any of the methods given above for the concomitants of a single ground form. In particular, the third method may be employed. Illustration is made with the quartic.

Thus
$$\{4\}\{3\} = \{7\}+\{61\}+\{52\}+\{43\}.$$
$\{8\}$ is not chosen as this belongs to $\{4\}\otimes\{2\}$. Instead choice is made of
$$\{71\}\to\{7\}+\{61\}, \quad \{53\}\to\{52\}+\{43\}.$$
Hence
$$\{4\}\otimes\{1^2\} = \{71\}+\{53\}.$$
Next
$$\begin{aligned}\{4\}\otimes\{1^2\}\{3\} &= [\{71\}+\{53\}]\{3\}\\ &= \{10\,.\,1\}+\{92\}+\{91^2\}+2\{83\}+\{821\}+2\{74\}+2\{731\}+\{65\}+\{641\}\\ &\quad+\{632\}+\{5^21\}+\{542\}+\{53^2\}.\end{aligned}$$
Thence
$$\begin{aligned}&\{10\,.\,1^2\}\to\{10\,.\,1\}+\{91^2\},\\ &\{93\}\to\{92\}+\{83\},\\ &\{831\}\to\{83\}+\{821\}+\{731\},\\ &\{75\}\to\{74\}+\{65\},\\ &\{741\}\to\{74\}+\{731\}+\{641\},\\ &\{633\}\to\{632\}+\{533\},\\ &\{552\}\to\{551\}+\{542\}.\end{aligned}$$
Hence
$$\{4\}\otimes\{1^3\} = \{10\,.\,1^2\}+\{93\}+\{831\}+\{75\}+\{741\}+\{63^2\}+\{5^22\}.$$

The application of alternating concomitant types is illustrated with one very simple example.

The equation to a line in three dimensions (four variables) is that of a linear complex $\{1^2\}$ whose second degree invariant $\{1^4\}$ is zero. Given three non-intersecting lines, the set of lines which intersect these three lines generate a quadric, the equation to which must be the vanishing of a concomitant of the three lines.

The concomitant must be either alternating or symmetric. The symmetric case is ruled out, for the symmetric concomitants are of type $\{p^2, q^2\}$ and do not include a quadric. It is therefore concluded that it is an alternating concomitant type.

Then
$$\{1^2\}\otimes\{1^3\} = \{31^3\}+\{2^3\}.$$
Clearly the first concomitant represents the point equation and the second concomitant the tangential equation of the given quadric.

The corresponding tableaux are
$$\begin{pmatrix}\alpha & \beta & \gamma\\ \alpha' & & \\ \beta' & & \\ \gamma' & & \end{pmatrix}\begin{pmatrix}\alpha & \beta\\ \alpha' & \gamma\\ \beta' & \gamma'\end{pmatrix}.$$

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References

Capelli, A. 1880 *Battaglini*, **18**, 17.
Capelli, A. 1882 *Mem. Accad. Lincei*, **12**, 1.
Cayley, A. 1889–98 *Collected Works*.
Clebsch, A. 1872 *Abh. Ges. Wiss. Göttingen*, **17**, 1.
Clebsch, A. & Capelli, A. 1891 *Mem. Accad. Lincei*, **7**, 161.
Clebsch, A. & Capelli, A. 1892 *Mem. Accad. Lincei*, (2) **1**, 3.
Clebsch, A. & Gordan, P. 1872 *Math. Ann.* **5**, 95.
Clebsch, A. & Gordan, P. 1875 *Math. Ann.* **6**, 436.
Eddington, A. S. 1923 *The mathematical theory of relativity*. Cambridge.
Eddington, A. S. 1936 *Relativity theory of protons and electrons*. Cambridge.
Gordan, P. 1903 *Math. Ann.* **56**, 1.
Grace, J. H. & Young, A. 1903 *The algebra of invariants*. Cambridge.
Hilbert, D. 1890 *Math. Ann.* **36**, 473.
Klein, F. 1921 *Gesammelte mathematische Abhandlungen*.
Levi-Civita, T. 1925 *The absolute differential calculus*. English ed. Glasgow.
Littlewood, D. E. 1936*a* *Proc. Lond. Math. Soc.* **40**, 49.
Littlewood, D. E. 1936*b* *J. Lond. Math. Soc.* **11**, 49.
Littlewood, D. E. 1940 *The theory of group characters and matrix representations of groups*. Oxford.
Littlewood, D. E. & Richardson, A. R. 1933 *Quart. J. Math.* **5**, 269.
Littlewood, D. E. & Richardson, A. R. 1934*a* *Quart. J. Math.* **6**, 184.
Littlewood, D. E. & Richardson, A. R. 1934*b* *Phil. Trans.* A, **233**, 99.
Meyer, W. F. 1892 *Berichte über den gegenwärtigen Stand der Invariantentheorie*.
Noether, E. 1908 *J. reine angew. Math.* **134**, 23.
Schur, I. 1901 *Ueber eine Klasse von Matrizen einer gegebene Matrix zuordnen lassen*. Berlin.
Schur, I. 1924*a* *S.B. preuss. Akad. Wiss.* 189.
Schur, I. 1924*b* *S.B. preuss. Akad. Wiss.* 297.
Schur, I. 1924*c* *S.B. preuss Akad. Wiss.* 346.
Sylvester, J. J. 1908 *Collected works*.
Turnbull, H. W. 1928 *The theory of determinants, matrices and invariants*. London and Glasgow.
Turnbull, H. W. & Young, A. 1926 *Trans. Camb. Phil. Soc.* **23**, 265.
Weitzenböck, R. 1923 *Invariantentheorie*.
Weyl, H. 1939 *The classical groups*. Princeton.
Young, A. 1901 *Proc. Lond. Math. Soc.* (1) **33**, 97.
Young, A. 1902 *Proc. Lond. Math. Soc.* (1) **34**, 361.
Young, A. 1924 *Proc. Lond. Math. Soc.* (2) **22**, 171.
Young, A. 1928 *Proc. Lond. Math. Soc.* (2) **28**, 255.
Young, A. 1930*a* *Proc. Lond. Math. Soc.* (2) **31**, 253.
Young, A. 1930*b* *Proc. Lond. Math. Soc.* (2) **31**, 273.
Young, A. 1932 *Proc. Lond. Math. Soc.* (2) **34**, 196.
Young, A. 1933 *Proc. Lond. Math. Soc.* (2) **35**, 425.
Young, A. 1934*a* *Proc. Lond. Math. Soc.* (2) **36**, 304.
Young, A. 1934*b* *Proc. Lond. Math. Soc.* (2) **37**, 441.
Young, A. 1935 *Phil. Trans.* A, **234**, 79.